1. Exploratory Data Analysis

This chapter presents the assumptions, principles, and techniques necessary to gain insight into data via EDA--exploratory data analysis.

1. Exploratory Data Analysis - Detailed Table of Contents [1.]

This chapter presents the assumptions, principles, and techniques necessary to gain insight into data via EDA--exploratory data analysis.

1.1. EDA Introduction

Summary

What is exploratory data analysis? How did it begin? How and where did it originate? How is it differentiated from other data analysis approaches, such as classical and Bayesian? Is EDA the same as statistical graphics? What role does statistical graphics play in EDA? Is statistical graphics identical to EDA?

These questions and related questions are dealt with in this section. This section answers these questions and provides the necessary frame of reference for EDA assumptions, principles, and techniques.

Table of Contents for Section 1

1.1.1. What is EDA?

Approach

Exploratory Data Analysis (EDA) is an approach/philosophy for data analysis that employs a variety of techniques (mostly graphical) to

1. maximize insight into a data set;
2. uncover underlying structure;
3. extract important variables;
4. detect outliers and anomalies;
5. test underlying assumptions;
6. develop parsimonious models; and
7. determine optimal factor settings.

Focus

The EDA approach is precisely that--an approach--not a set of techniques, but an attitude/philosophy about how a data analysis should be carried out.

Philosophy

EDA is not identical to statistical graphics although the two terms are used almost interchangeably. Statistical graphics is a collection of techniques--all graphically based and all focusing on one data characterization aspect. EDA encompasses a larger venue; EDA is an approach to data analysis that postpones the usual assumptions about what kind of model the data follow with the more direct approach of allowing the data itself to reveal its underlying structure and model. EDA is not a mere collection of techniques; EDA is a philosophy as to how we dissect a data set; what we look for; how we look; and how we interpret. It is true that EDA heavily uses the collection of techniques that we call "statistical graphics", but it is not identical to statistical graphics per se.

History

The seminal work in EDA is Exploratory Data Analysis, Tukey, (1977). Over the years it has benefitted from other noteworthy publications such as Data Analysis and Regression, Mosteller and Tukey (1977), Interactive Data Analysis, Hoaglin (1977), The ABC's of EDA, Velleman and Hoaglin (1981) and has gained a large following as "the" way to analyze a data set.

Techniques

Most EDA techniques are graphical in nature with a few quantitative techniques. The reason for the heavy reliance on graphics is that by its very nature the main role of EDA is to open-mindedly explore, and graphics gives the analysts unparalleled power to do so, enticing the data to reveal its structural secrets, and being always ready to gain some new, often unsuspected, insight into the data. In combination with the natural pattern-recognition capabilities that we all possess, graphics provides, of course, unparalleled power to carry this out.

The particular graphical techniques employed in EDA are often quite simple, consisting of various techniques of:

1. Plotting the raw data (such as data traces, histograms, bihistograms, probability plots, lag plots, block plots, and Youden plots.

2. Plotting simple statistics such as mean plots, standard deviation plots, box plots, and main effects plots of the raw data.

3. Positioning such plots so as to maximize our natural pattern-recognition abilities, such as using multiple plots per page.

1.1.2. How Does Exploratory Data Analysis differ from Classical Data Analysis?

Data Analysis Approaches

EDA is a data analysis approach. What other data analysis approaches exist and how does EDA differ from these other approaches? Three popular data analysis approaches are:

1. Classical
2. Exploratory (EDA)
3. Bayesian

Paradigms for Analysis Techniques

These three approaches are similar in that they all start with a general science/engineering problem and all yield science/engineering conclusions. The difference is the sequence and focus of the intermediate steps.

For classical analysis, the sequence is

> Problem => Data => Model => Analysis => Conclusions

For EDA, the sequence is

> Problem => Data => Analysis => Model => Conclusions

For Bayesian, the sequence is

> Problem => Data => Model => Prior Distribution => Analysis => Conclusions

Method of dealing with underlying model for the data distinguishes the 3 approaches

Thus for classical analysis, the data collection is followed by the imposition of a model (normality, linearity, etc.) and the analysis, estimation, and testing that follows are focused on the parameters of that model. For EDA, the data collection is not followed by a model imposition; rather it is followed immediately by analysis with a goal of inferring what model would be appropriate. Finally, for a Bayesian analysis, the analyst attempts to incorporate scientific/engineering knowledge/expertise into the analysis by imposing a data-independent distribution on the parameters of the selected model; the analysis thus consists of formally combining both the prior distribution on the parameters and the collected data to jointly make inferences and/or test assumptions about the model parameters.

In the real world, data analysts freely mix elements of all of the above three approaches (and other approaches). The above distinctions were made to emphasize the major differences among the three approaches.

Further discussion of the distinction between the classical and EDA approaches

Focusing on EDA versus classical, these two approaches differ as follows:

1. Models
2. Focus
3. Techniques
4. Rigor
5. Data Treatment
6. Assumptions

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.1.2.1. Model

Classical	The classical approach imposes models (both deterministic and probabilistic) on the data. Deterministic models include, for example, regression models and analysis of variance (ANOVA) models. The most common probabilistic model assumes that the errors about the deterministic model are normally distributed--this assumption affects the validity of the ANOVA F tests.
Exploratory	The Exploratory Data Analysis approach does not impose deterministic or probabilistic models on the data. On the contrary, the EDA approach allows the data to suggest admissible models that best fit the data.

1.1.2.2. Focus

Classical	The two approaches differ substantially in focus. For classical analysis, the focus is on the model--estimating parameters of the model and generating predicted values from the model.
Exploratory	For exploratory data analysis, the focus is on the data--its structure, outliers, and models suggested by the data.

1.1.2.3. Techniques

Classical	Classical techniques are generally quantitative in nature. They include ANOVA, t tests, chi-squared tests, and F tests.
Exploratory	EDA techniques are generally graphical. They include scatter plots, character plots, box plots, histograms, bihistograms, probability plots, residual plots, and mean plots.

1.1.2.4. Rigor

Classical	Classical techniques serve as the probabilistic foundation of science and engineering; the most important characteristic of classical techniques is that they are rigorous, formal, and "objective".
Exploratory	EDA techniques do not share in that rigor or formality. EDA techniques make up for that lack of rigor by being very suggestive, indicative, and insightful about what the appropriate model should be.
	EDA techniques are subjective and depend on interpretation which may differ from analyst to analyst, although experienced analysts commonly arrive at identical conclusions.

1.1.2.5. Data Treatment

Classical	Classical estimation techniques have the characteristic of taking all of the data and mapping the data into a few numbers ("estimates"). This is both a virtue and a vice. The virtue is that these few numbers focus on important characteristics (location, variation, etc.) of the population. The vice is that concentrating on these few characteristics can filter out other characteristics (skewness, tail length, autocorrelation, etc.) of the same population. In this sense there is a loss of information due to this "filtering" process.
Exploratory	The EDA approach, on the other hand, often makes use of (and shows) all of the available data. In this sense there is no corresponding loss of information.

1.1.2.6. Assumptions

Classical The "good news" of the classical approach is that tests based on classical techniques are usually very sensitive--that is, if a true shift in location, say, has occurred, such tests frequently have the power to detect such a shift and to conclude that such a shift is "statistically significant". The "bad news" is that classical tests depend on underlying assumptions (e.g., normality), and hence the validity of the test conclusions becomes dependent on the validity of the underlying assumptions. Worse yet, the exact underlying assumptions may be unknown to the analyst, or if known, untested. Thus the validity of the scientific conclusions becomes intrinsically linked to the validity of the underlying assumptions. In practice, if such assumptions are unknown or untested, the validity of the scientific conclusions becomes suspect.

Exploratory Many EDA techniques make little or no assumptions--they present and show the data--all of the data--as is, with fewer encumbering assumptions.

1.1.3. How Does Exploratory Data Analysis Differ from Summary Analysis?

Summary

A summary analysis is simply a numeric reduction of a historical data set. It is quite passive. Its focus is in the past. Quite commonly, its purpose is to simply arrive at a few key statistics (for example, mean and standard deviation) which may then either replace the data set or be added to the data set in the form of a summary table.

Exploratory

In contrast, EDA has as its broadest goal the desire to gain insight into the engineering/scientific process behind the data. Whereas summary statistics are passive and historical, EDA is active and futuristic. In an attempt to "understand" the process and improve it in the future, EDA uses the data as a "window" to peer into the heart of the process that generated the data. There is an archival role in the research and manufacturing world for summary statistics, but there is an enormously larger role for the EDA approach.

1.1.4. What are the EDA Goals?

Primary and Secondary Goals

The primary goal of EDA is to maximize the analyst's insight into a data set and into the underlying structure of a data set, while providing all of the specific items that an analyst would want to extract from a data set, such as:

1. a good-fitting, parsimonious model
2. a list of outliers
3. a sense of robustness of conclusions
4. estimates for parameters
5. uncertainties for those estimates
6. a ranked list of important factors
7. conclusions as to whether individual factors are statistically significant
8. optimal settings

Insight into the Data

Insight implies detecting and uncovering underlying structure in the data. Such underlying structure may not be encapsulated in the list of items above; such items serve as the specific targets of an analysis, but the real insight and "feel" for a data set comes as the analyst judiciously probes and explores the various subtleties of the data. The "feel" for the data comes almost exclusively from the application of various graphical techniques, the collection of which serves as the window into the essence of the data. Graphics are irreplaceable--there are no quantitative analogues that will give the same insight as well-chosen graphics.

To get a "feel" for the data, it is not enough for the analyst to know what is in the data; the analyst also must know what is not in the data, and the only way to do that is to draw on our own human pattern-recognition and comparative abilities in the context of a series of judicious graphical techniques applied to the data.

1.1.5. The Role of Graphics

Quantitative/ Graphical

Statistics and data analysis procedures can broadly be split into two parts:

- quantitative
- graphical

Quantitative

Quantitative techniques are the set of statistical procedures that yield numeric or tabular output. Examples of quantitative techniques include:

- hypothesis testing
- analysis of variance
- point estimates and confidence intervals
- least squares regression

These and similar techniques are all valuable and are mainstream in terms of classical analysis.

Graphical

On the other hand, there is a large collection of statistical tools that we generally refer to as graphical techniques. These include:

- scatter plots
- histograms
- probability plots
- residual plots
- box plots
- block plots

EDA Approach Relies Heavily on Graphical Techniques

The EDA approach relies heavily on these and similar graphical techniques. Graphical procedures are not just tools that we could use in an EDA context, they are tools that we must use. Such graphical tools are the shortest path to gaining insight into a data set in terms of

- testing assumptions
- model selection
- model validation
- estimator selection
- relationship identification
- factor effect determination

- outlier detection

If one is not using statistical graphics, then one is forfeiting insight into one or more aspects of the underlying structure of the data.

1.1.6. An EDA/Graphics Example

Anscombe Example

A simple, classic (Anscombe) example of the central role that graphics play in terms of providing insight into a data set starts with the following data set:

Data

```
  X              Y
10.00           8.04
 8.00           6.95
13.00           7.58
 9.00           8.81
11.00           8.33
14.00           9.96
 6.00           7.24
 4.00           4.26
12.00          10.84
 7.00           4.82
 5.00           5.68
```

Summary Statistics

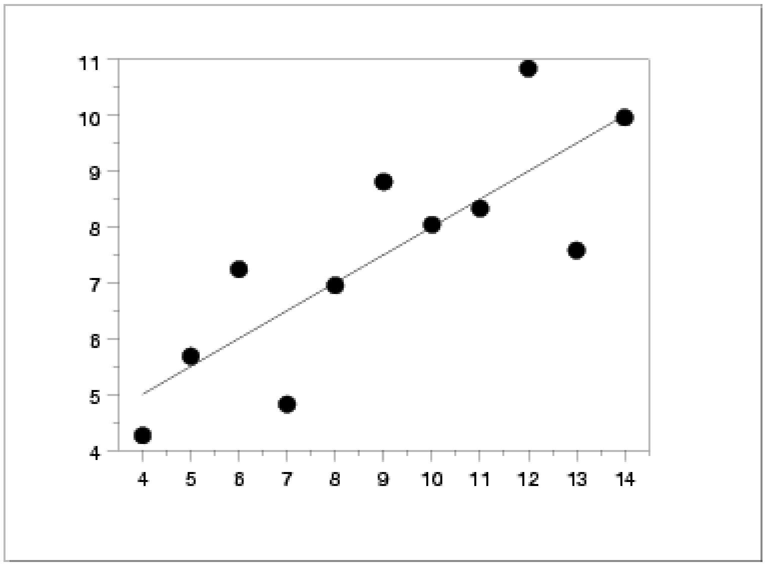
If the goal of the analysis is to compute summary statistics plus determine the best linear fit for Y as a function of X, the results might be given as:

N = 11
Mean of X = 9.0
Mean of Y = 7.5
Intercept = 3
Slope = 0.5
Residual standard deviation = 1.237
Correlation = 0.816

The above quantitative analysis, although valuable, gives us only limited insight into the data.

Scatter Plot

In contrast, the following simple scatter plot of the data



suggests the following:

1. The data set "behaves like" a linear curve with some scatter;
2. there is no justification for a more complicated model (e.g., quadratic);
3. there are no outliers;
4. the vertical spread of the data appears to be of equal height irrespective of the *X*-value; this indicates that the data are equally-precise throughout and so a "regular" (that is, equi-weighted) fit is appropriate.

Three Additional Data Sets

This kind of characterization for the data serves as the core for getting insight/feel for the data. Such insight/feel does not come from the quantitative statistics; on the contrary, calculations of quantitative statistics such as intercept and slope should be subsequent to the characterization and will make sense only if the characterization is true. To illustrate the loss of information that results when the graphics insight step is skipped, consider the following three data sets [Anscombe data sets 2, 3, and 4]:

X2	Y2	X3	Y3	X4	Y4
10.00	9.14	10.00	7.46	8.00	6.58
8.00	8.14	8.00	6.77	8.00	5.76
13.00	8.74	13.00	12.74	8.00	7.71
9.00	8.77	9.00	7.11	8.00	8.84
11.00	9.26	11.00	7.81	8.00	8.47
14.00	8.10	14.00	8.84	8.00	7.04
6.00	6.13	6.00	6.08	8.00	5.25
4.00	3.10	4.00	5.39	19.00	12.50
12.00	9.13	12.00	8.15	8.00	5.56
7.00	7.26	7.00	6.42	8.00	7.91
5.00	4.74	5.00	5.73	8.00	6.89

Quantitative Statistics for Data Set 2

A quantitative analysis on data set 2 yields

N = 11
Mean of X = 9.0
Mean of Y = 7.5
Intercept = 3

Slope = 0.5
Residual standard deviation = 1.237
Correlation = 0.816

which is identical to the analysis for data set 1. One might naively assume that the two data sets are "equivalent" since that is what the statistics tell us; but what do the statistics not tell us?

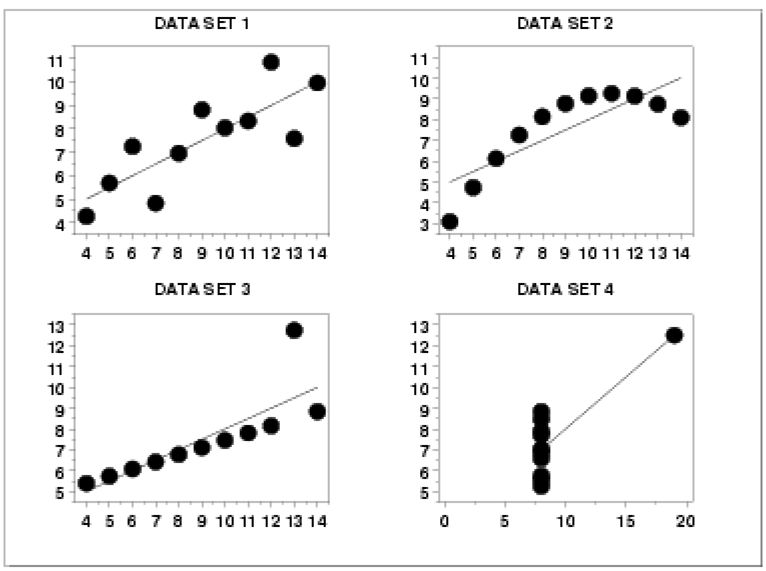
Quantitative Statistics for Data Sets 3 and 4

Remarkably, a quantitative analysis on data sets 3 and 4 also yields

N = 11
Mean of X = 9.0
Mean of Y = 7.5
Intercept = 3
Slope = 0.5
Residual standard deviation = 1.236
Correlation = 0.816 (0.817 for data set 4)

which implies that in some quantitative sense, all four of the data sets are "equivalent". In fact, the four data sets are far from "equivalent" and a scatter plot of each data set, which would be step 1 of any EDA approach, would tell us that immediately.

Scatter Plots



Interpretation of Scatter Plots

Conclusions from the scatter plots are:

1. data set 1 is clearly linear with some scatter.
2. data set 2 is clearly quadratic.
3. data set 3 clearly has an outlier.
4. data set 4 is obviously the victim of a poor experimental design with a single point far removed from the bulk of the data "wagging the dog".

Importance

These points are exactly the substance that provide and *of Exploratory Analysis*

define "insight" and "feel" for a data set. They are the goals and the fruits of an open exploratory data analysis (EDA) approach to the data. Quantitative statistics are not wrong per se, but they are incomplete. They are incomplete because they are numeric **summaries** which in the summarization operation do a good job of focusing on a particular aspect of the data (e.g., location, intercept, slope, degree of relatedness, etc.) by judiciously reducing the data to a few numbers. Doing so also **filters** the data, necessarily omitting and screening out other sometimes crucial information in the focusing operation. Quantitative statistics focus but also filter; and filtering is exactly what makes the quantitative approach incomplete at best and misleading at worst.

The estimated intercepts (= 3) and slopes (= 0.5) for data sets 2, 3, and 4 are misleading because the estimation is done in the context of an assumed linear model and that linearity assumption is the fatal flaw in this analysis.

The EDA approach of deliberately postponing the model selection until further along in the analysis has many rewards, not the least of which is the ultimate convergence to a much-improved model and the formulation of valid and supportable scientific and engineering conclusions.

1.1.7. General Problem Categories

Problem Classification

The following table is a convenient way to classify EDA problems.

Univariate and Control

UNIVARIATE	CONTROL
Data: A single column of numbers, *Y*. Model: y = constant + error Output: 1. A number (the estimated constant in the model). 2. An estimate of uncertainty for the constant. 3. An estimate of the distribution for the error. Techniques: • 4-Plot • Probability Plot • PPCC Plot	Data: A single column of numbers, *Y*. Model: y = constant + error Output: A "yes" or "no" to the question "Is the system out of control?". Techniques: • Control Charts

Comparative and Screening

COMPARATIVE	SCREENING
Data: A single response variable and k independent variables (Y, X_1, X_2, ... , X_k), primary focus is on	Data: A single response variable and k independent variables (Y, X_1, X_2, ... , X_k).

	one (the primary factor) of these independent variables. Model: $y = f(x_1, x_2, ..., x_k)$ + error Output: A "yes" or "no" to the question "Is the primary factor significant?". Techniques: • Block Plot • Scatter Plot • Box Plot	Model: $y = f(x_1, x_2, ..., x_k)$ + error Output: 1. A ranked list (from most important to least important) of factors. 2. Best settings for the factors. 3. A good model/prediction equation relating Y to the factors. Techniques: • Block Plot • Probability Plot • Bihistogram
Optimization and Regression	OPTIMIZATION Data: A single response variable and k independent variables ($Y, X_1, X_2, ... , X_k$). Model: $y = f(x_1, x_2, ..., x_k)$ + error Output: Best settings for the factor variables. Techniques: • Block Plot • Least Squares Fitting • Contour Plot	REGRESSION Data: A single response variable and k independent variables ($Y, X_1, X_2, ... , X_k$). The independent variables can be continuous. Model: $y = f(x_1, x_2, ..., x_k)$ + error Output: A good model/prediction equation relating Y to the factors. Techniques: • Least Squares Fitting • Scatter Plot

	6-Plot

Time Series and Multivariate

TIME SERIES	MULTIVARIATE
Data: A column of time dependent numbers, Y. In addition, time is an indpendent variable. The time variable can be either explicit or implied. If the data are not equi-spaced, the time variable should be explicitly provided. Model: $y_t = f(t)$ + error The model can be either a time domain based or frequency domain based. Output: A good model/prediction equation relating Y to previous values of Y. Techniques: • Autocorrelation Plot • Spectrum • Complex Demodulation Amplitude Plot • Complex Demodulation Phase Plot • ARIMA Models	Data: k factor variables (X_1, X_2, ... , X_k). Model: The model is not explicit. Output: Identify underlying correlation structure in the data. Techniques: • Star Plot • Scatter Plot Matrix • Conditioning Plot • Profile Plot • Principal Components • Clustering • Discrimination/Classification Note that multivarate analysis is only covered lightly in this Handbook.

1.2. EDA Assumptions

Summary

The gamut of scientific and engineering experimentation is virtually limitless. In this sea of diversity is there any common basis that allows the analyst to systematically and validly arrive at supportable, repeatable research conclusions?

Fortunately, there is such a basis and it is rooted in the fact that every measurement process, however complicated, has certain underlying assumptions. This section deals with what those assumptions are, why they are important, how to go about testing them, and what the consequences are if the assumptions do not hold.

Table of Contents for Section 2

1.2.1. Underlying Assumptions

Assumptions Underlying a Measurement Process

There are four assumptions that typically underlie all measurement processes; namely, that the data from the process at hand "behave like":

1. random drawings;
2. from a fixed distribution;
3. with the distribution having fixed location; and
4. with the distribution having fixed variation.

Univariate or Single Response Variable

The "fixed location" referred to in item 3 above differs for different problem types. The simplest problem type is univariate; that is, a single variable. For the univariate problem, the general model

response = deterministic component + random component

becomes

response = constant + error

Assumptions for Univariate Model

For this case, the "fixed location" is simply the unknown constant. We can thus imagine the process at hand to be operating under constant conditions that produce a single column of data with the properties that

- the data are uncorrelated with one another;
- the random component has a fixed distribution;
- the deterministic component consists of only a constant; and
- the random component has fixed variation.

Extrapolation to a Function of Many Variables

The universal power and importance of the univariate model is that it can easily be extended to the more general case where the deterministic component is not just a constant, but is in fact a function of many variables, and the engineering objective is to characterize and model the function.

Residuals

The key point is that regardless of how many factors there

Will Behave According to Univariate Assumptions

are, and regardless of how complicated the function is, if the engineer succeeds in choosing a good model, then the differences (residuals) between the raw response data and the predicted values from the fitted model should themselves behave like a univariate process. Furthermore, the residuals from this univariate process fit will behave like:

- random drawings;
- from a fixed distribution;
- with fixed location (namely, 0 in this case); and
- with fixed variation.

Validation of Model

Thus if the residuals from the fitted model do in fact behave like the ideal, then testing of underlying assumptions becomes a tool for the validation and quality of fit of the chosen model. On the other hand, if the residuals from the chosen fitted model violate one or more of the above univariate assumptions, then the chosen fitted model is inadequate and an opportunity exists for arriving at an improved model.

1.2.2. Importance

Predictability and Statistical Control	Predictability is an all-important goal in science and engineering. If the four underlying assumptions hold, then we have achieved probabilistic predictability--the ability to make probability statements not only about the process in the past, but also about the process in the future. In short, such processes are said to be "in statistical control".
Validity of Engineering Conclusions	Moreover, if the four assumptions are valid, then the process is amenable to the generation of valid scientific and engineering conclusions. If the four assumptions are not valid, then the process is drifting (with respect to location, variation, or distribution), unpredictable, and out of control. A simple characterization of such processes by a location estimate, a variation estimate, or a distribution "estimate" inevitably leads to engineering conclusions that are not valid, are not supportable (scientifically or legally), and which are not repeatable in the laboratory.

1.2.3. Techniques for Testing Assumptions

Testing Underlying Assumptions Helps Assure the Validity of Scientific and Engineering Conclusions

Because the validity of the final scientific/engineering conclusions is inextricably linked to the validity of the underlying univariate assumptions, it naturally follows that there is a real necessity that each and every one of the above four assumptions be routinely tested.

Four Techniques to Test Underlying Assumptions

The following EDA techniques are simple, efficient, and powerful for the routine testing of underlying assumptions:

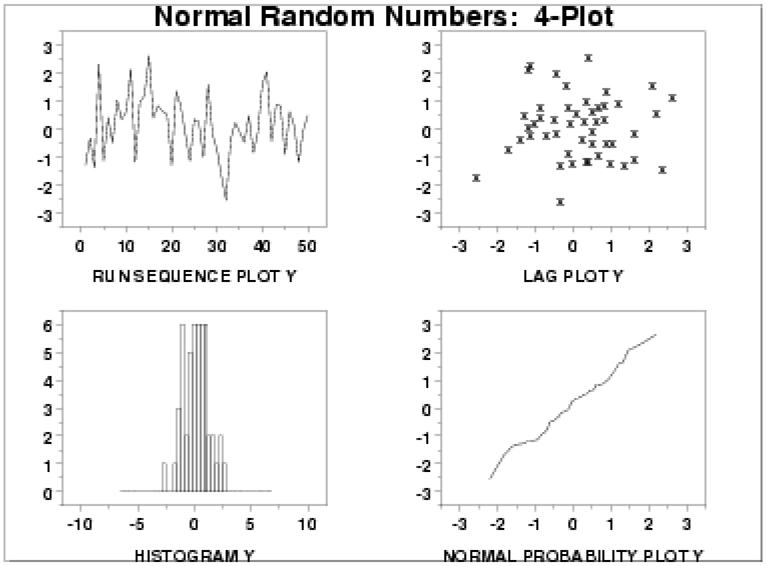
1. run sequence plot (Y_i versus i)
2. lag plot (Y_i versus Y_{i-1})
3. histogram (counts versus subgroups of Y)
4. normal probability plot (ordered Y versus theoretical ordered Y)

Plot on a Single Page for a Quick Characterization of the Data

The four EDA plots can be juxtaposed for a quick look at the characteristics of the data. The plots below are ordered as follows:

1. Run sequence plot - upper left
2. Lag plot - upper right
3. Histogram - lower left
4. Normal probability plot - lower right

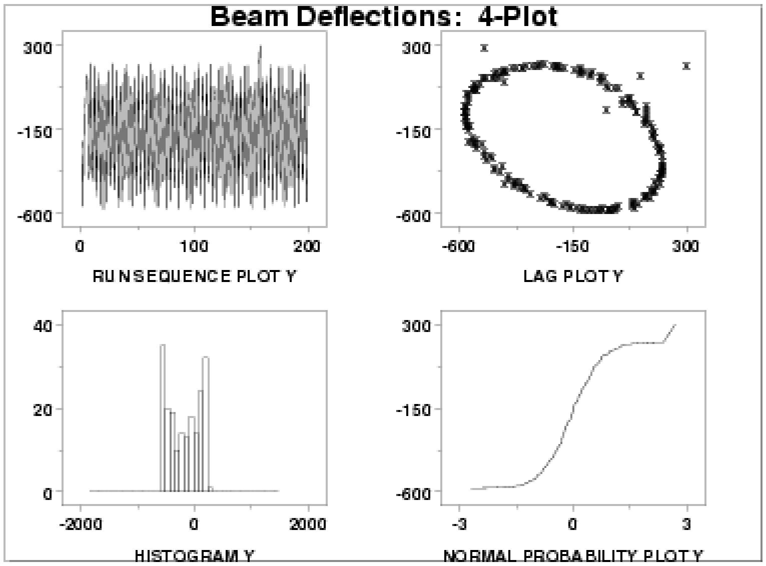
Sample Plot: Assumptions Hold



This [4-plot](#) reveals a process that has fixed location, fixed variation, is random, apparently has a fixed approximately normal distribution, and has no outliers.

Sample Plot: Assumptions Do Not Hold

If one or more of the four underlying assumptions do not hold, then it will show up in the various plots as demonstrated in the following example.



This [4-plot](#) reveals a process that has fixed location, fixed variation, is non-random (oscillatory), has a non-normal, U-shaped distribution, and has several outliers.

1.2.4. Interpretation of 4-Plot

Interpretation of EDA Plots: Flat and Equi-Banded, Random, Bell-Shaped, and Linear

The four EDA plots discussed on the previous page are used to test the underlying assumptions:

1. **Fixed Location:**
   If the fixed location assumption holds, then the run sequence plot will be flat and non-drifting.
2. **Fixed Variation:**
   If the fixed variation assumption holds, then the vertical spread in the run sequence plot will be the approximately the same over the entire horizontal axis.
3. **Randomness:**
   If the randomness assumption holds, then the lag plot will be structureless and random.
4. **Fixed Distribution:**
   If the fixed distribution assumption holds, in particular if the fixed normal distribution holds, then
   1. the histogram will be bell-shaped, and
   2. the normal probability plot will be linear.

Plots Utilized to Test the Assumptions

Conversely, the underlying assumptions are tested using the EDA plots:

- **Run Sequence Plot:**
  If the run sequence plot is flat and non-drifting, the fixed-location assumption holds. If the run sequence plot has a vertical spread that is about the same over the entire plot, then the fixed-variation assumption holds.
- **Lag Plot:**
  If the lag plot is structureless, then the randomness assumption holds.
- **Histogram:**
  If the histogram is bell-shaped, the underlying distribution is symmetric and perhaps approximately normal.

Normal Probability Plot:
If the normal probability plot is linear, the underlying distribution is approximately normal.

If all four of the assumptions hold, then the process is said definitionally to be "in statistical control".

1.2.5. Consequences

What If Assumptions Do Not Hold?	If some of the underlying assumptions do not hold, what can be done about it? What corrective actions can be taken? The positive way of approaching this is to view the testing of underlying assumptions as a framework for learning about the process. Assumption-testing promotes insight into important aspects of the process that may not have surfaced otherwise.
Primary Goal is Correct and Valid Scientific Conclusions	The primary goal is to have correct, validated, and complete scientific/engineering conclusions flowing from the analysis. This usually includes intermediate goals such as the derivation of a good-fitting model and the computation of realistic parameter estimates. It should always include the ultimate goal of an understanding and a "feel" for "what makes the process tick". There is no more powerful catalyst for discovery than the bringing together of an experienced/expert scientist/engineer and a data set ripe with intriguing "anomalies" and characteristics.
Consequences of Invalid Assumptions	The following sections discuss in more detail the consequences of invalid assumptions: 1. Consequences of non-randomness 2. Consequences of non-fixed location parameter 3. Consequences of non-fixed variation 4. Consequences related to distributional assumptions

1.2.5.1. Consequences of Non-Randomness

Randomness Assumption

There are four underlying assumptions:

1. randomness;
2. fixed location;
3. fixed variation; and
4. fixed distribution.

The randomness assumption is the most critical but the least tested.

Consequeces of Non-Randomness

If the randomness assumption does not hold, then

1. All of the usual statistical tests are invalid.
2. The calculated uncertainties for commonly used statistics become meaningless.
3. The calculated minimal sample size required for a pre-specified tolerance becomes meaningless.
4. The simple model: y = constant + error becomes invalid.
5. The parameter estimates become suspect and non-supportable.

Non-Randomness Due to Autocorrelation

One specific and common type of non-randomness is autocorrelation. Autocorrelation is the correlation between Y_t and Y_{t-k}, where k is an integer that defines the lag for the autocorrelation. That is, autocorrelation is a time dependent non-randomness. This means that the value of the current point is highly dependent on the previous point if $k = 1$ (or k points ago if k is not 1). Autocorrelation is typically detected via an autocorrelation plot or a lag plot.

If the data are not random due to autocorrelation, then

1. Adjacent data values may be related.
2. There may not be n independent snapshots of the phenomenon under study.
3. There may be undetected "junk"-outliers.
4. There may be undetected "information-rich"-outliers.

1.2.5.2. Consequences of Non-Fixed Location Parameter

Location Estimate

The usual estimate of location is the mean

$$\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

from N measurements $Y_1, Y_2, \ldots, Y_N$.

Consequences of Non-Fixed Location

If the run sequence plot does not support the assumption of fixed location, then

1. The location may be drifting.
2. The single location estimate may be meaningless (if the process is drifting).
3. The choice of location estimator (e.g., the sample mean) may be sub-optimal.
4. The usual formula for the uncertainty of the mean:

   $$s(\bar{Y}) = \frac{1}{\sqrt{N(N-1)}}\sqrt{\sum_{i=1}^{N}(Y_i - \bar{Y})^2}$$

   may be invalid and the numerical value optimistically small.
5. The location estimate may be poor.
6. The location estimate may be biased.

1.2.5.3. Consequences of Non-Fixed Variation Parameter

Variation Estimate

The usual estimate of variation is the standard deviation

$$s_Y = \frac{1}{\sqrt{(N-1)}}\sqrt{\sum_{i=1}^{N}(Y_i - \bar{Y})^2}$$

from N measurements Y_1, Y_2, ... , Y_N.

Consequences of Non-Fixed Variation

If the run sequence plot does not support the assumption of fixed variation, then

1. The variation may be drifting.
2. The single variation estimate may be meaningless (if the process variation is drifting).
3. The variation estimate may be poor.
4. The variation estimate may be biased.

1.2.5.4. Consequences Related to Distributional Assumptions

Distributional Analysis

Scientists and engineers routinely use the mean (average) to estimate the "middle" of a distribution. It is not so well known that the variability and the noisiness of the mean as a location estimator are intrinsically linked with the underlying distribution of the data. For certain distributions, the mean is a poor choice. For any given distribution, there exists an optimal choice-- that is, the estimator with minimum variability/noisiness. This optimal choice may be, for example, the median, the midrange, the midmean, the mean, or something else. The implication of this is to "estimate" the distribution first, and then--based on the distribution--choose the optimal estimator. The resulting engineering parameter estimators will have less variability than if this approach is not followed.

Case Studies

The airplane glass failure case study gives an example of determining an appropriate distribution and estimating the parameters of that distribution. The uniform random numbers case study gives an example of determining a more appropriate centrality parameter for a non-normal distribution.

Other consequences that flow from problems with distributional assumptions are:

Distribution

1. The distribution may be changing.
2. The single distribution estimate may be meaningless (if the process distribution is changing).
3. The distribution may be markedly non-normal.
4. The distribution may be unknown.
5. The true probability distribution for the error may remain unknown.

Model

1. The model may be changing.
2. The single model estimate may be meaningless.
3. The default model

   Y = constant + error

   may be invalid.
4. If the default model is insufficient, information about

a better model may remain undetected.
5. A poor deterministic model may be fit.
6. Information about an improved model may go undetected.

Process

1. The process may be out-of-control.
2. The process may be unpredictable.
3. The process may be un-modelable.

1.3. EDA Techniques

Summary

After you have collected a set of data, how do you do an exploratory data analysis? What techniques do you employ? What do the various techniques focus on? What conclusions can you expect to reach?

This section provides answers to these kinds of questions via a gallery of EDA techniques and a detailed description of each technique. The techniques are divided into graphical and quantitative techniques. For exploratory data analysis, the emphasis is primarily on the graphical techniques.

Table of Contents for Section 3

1.3.1. Introduction

Graphical and Quantitative Techniques	This section describes many techniques that are commonly used in exploratory and classical data analysis. This list is by no means meant to be exhaustive. Additional techniques (both graphical and quantitative) are discussed in the other chapters. Specifically, the product comparisons chapter has a much more detailed description of many classical statistical techniques. EDA emphasizes graphical techniques while classical techniques emphasize quantitative techniques. In practice, an analyst typically uses a mixture of graphical and quantitative techniques. In this section, we have divided the descriptions into graphical and quantitative techniques. This is for organizational clarity and is not meant to discourage the use of both graphical and quantitiative techniques when analyzing data.
Use of Techniques Shown in Case Studies	This section emphasizes the techniques themselves; how the graph or test is defined, published references, and sample output. The use of the techniques to answer engineering questions is demonstrated in the case studies section. The case studies do not demonstrate all of the techniques.
Availability in Software	The sample plots and output in this section were generated with the Dataplot software program. Other general purpose statistical data analysis programs can generate most of the plots, intervals, and tests discussed here, or macros can be written to acheive the same result.

1.3.2. Analysis Questions

EDA Questions

Some common questions that exploratory data analysis is used to answer are:

1. What is a typical value?
2. What is the uncertainty for a typical value?
3. What is a good distributional fit for a set of numbers?
4. What is a percentile?
5. Does an engineering modification have an effect?
6. Does a factor have an effect?
7. What are the most important factors?
8. Are measurements coming from different laboratories equivalent?
9. What is the best function for relating a response variable to a set of factor variables?
10. What are the best settings for factors?
11. Can we separate signal from noise in time dependent data?
12. Can we extract any structure from multivariate data?
13. Does the data have outliers?

Analyst Should Identify Relevant Questions for his Engineering Problem

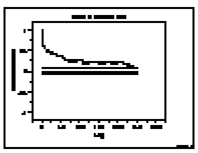
A critical early step in any analysis is to identify (for the engineering problem at hand) which of the above questions are relevant. That is, we need to identify which questions we want answered and which questions have no bearing on the problem at hand. After collecting such a set of questions, an equally important step, which is invaluable for maintaining focus, is to prioritize those questions in decreasing order of importance. EDA techniques are tied in with each of the questions. There are some EDA techniques (e.g., the scatter plot) that are broad-brushed and apply almost universally. On the other hand, there are a large number of EDA techniques that are specific and whose specificity is tied in with one of the above questions. Clearly if one chooses not to explicitly identify relevant questions, then one cannot take advantage of these question-specific EDA technqiues.

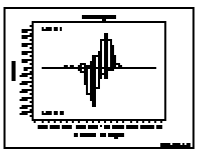
EDA Approach Emphasizes Graphics

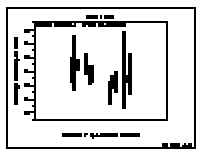
Most of these questions can be addressed by techniques discussed in this chapter. The process modeling and process improvement chapters also address many of the questions above. These questions are also relevant for the classical approach to statistics. What distinguishes the EDA approach is an emphasis on graphical techniques to gain insight as opposed to the classical approach of quantitative tests. Most data analysts will use a mix of graphical and classical quantitative techniques to address these problems.

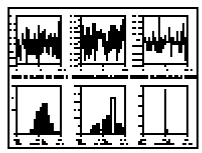
1.3.3. Graphical Techniques: Alphabetic

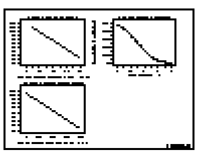
This section provides a gallery of some useful graphical techniques. The techniques are ordered alphabetically, so this section is not intended to be read in a sequential fashion. The use of most of these graphical techniques is demonstrated in the case studies in this chapter. A few of these graphical techniques are demonstrated in later chapters.

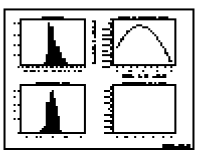

Autocorrelation Plot: 1.3.3.1

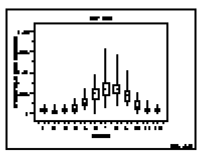

Bihistogram: 1.3.3.2

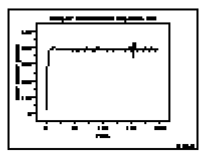

Block Plot: 1.3.3.3

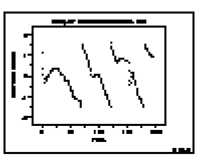

Bootstrap Plot: 1.3.3.4

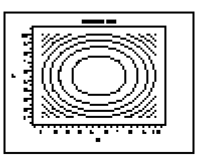

Box-Cox Linearity Plot: 1.3.3.5

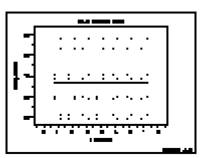

Box-Cox Normality Plot: 1.3.3.6

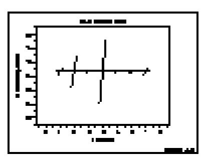

Box Plot: 1.3.3.7

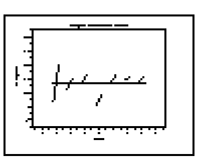

Complex Demodulation Amplitude Plot: 1.3.3.8

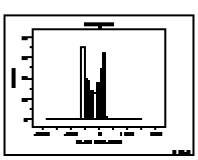

Complex Demodulation Phase Plot: 1.3.3.9

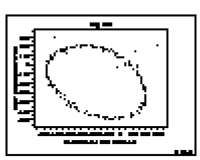

Contour Plot: 1.3.3.10

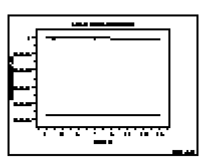

DOE Scatter Plot: 1.3.3.11

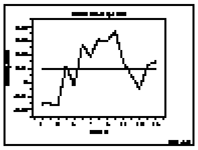

DOE Mean Plot: 1.3.3.12

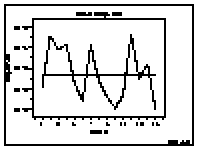

DOE Standard Deviation Plot: 1.3.3.13

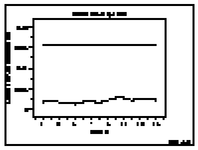

Histogram: 1.3.3.14

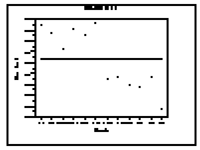

Lag Plot: 1.3.3.15

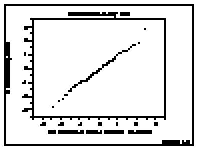

Linear Correlation Plot: 1.3.3.16

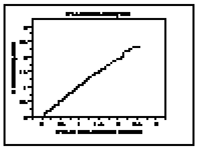

Linear Intercept Plot: 1.3.3.17

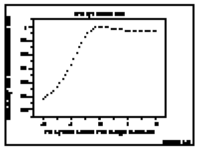

Linear Slope Plot: 1.3.3.18

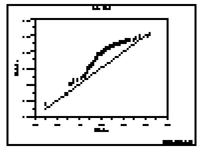

Linear Residual Standard Deviation Plot: 1.3.3.19

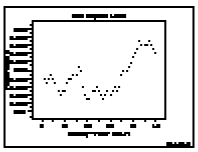

Mean Plot: 1.3.3.20

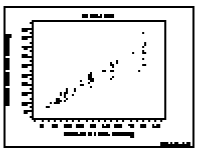

Normal Probability Plot: 1.3.3.21

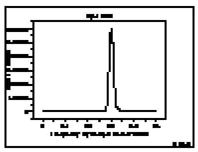

Probability Plot: 1.3.3.22

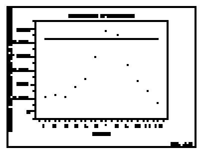

Probability Plot Correlation Coefficient Plot: 1.3.3.23

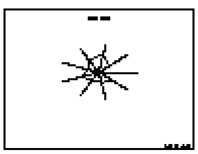

Quantile-Quantile Plot: 1.3.3.24

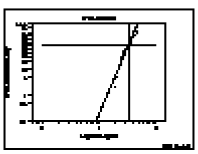

Run Sequence Plot: 1.3.3.25

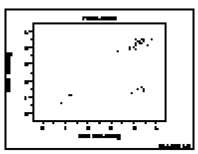

Scatter Plot: 1.3.3.26

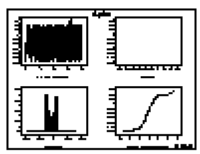

Spectrum: 1.3.3.27

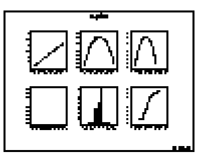

Standard Deviation Plot: 1.3.3.28


Star Plot: 1.3.3.29


Weibull Plot: 1.3.3.30


Youden Plot: 1.3.3.31


4-Plot: 1.3.3.32


6-Plot: 1.3.3.33

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

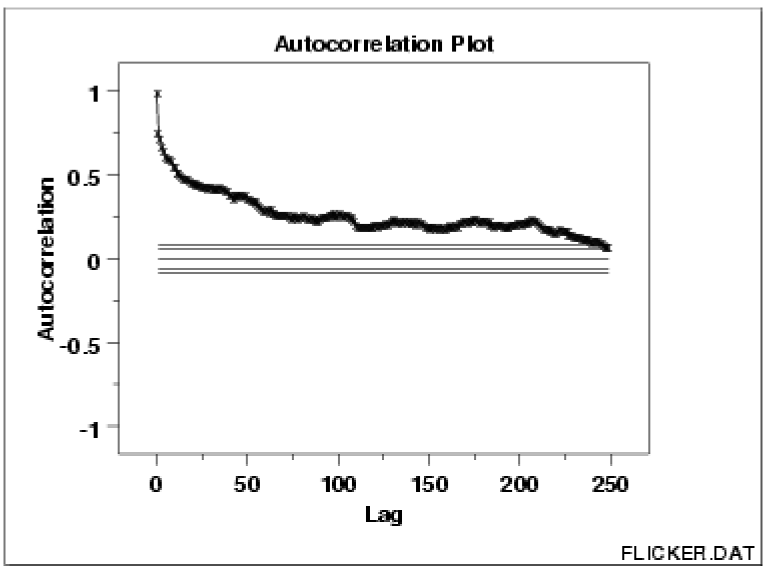
1.3.3.1. Autocorrelation Plot

Purpose: Check Randomness

Autocorrelation plots (Box and Jenkins, pp. 28-32) are a commonly-used tool for checking randomness in a data set. This randomness is ascertained by computing autocorrelations for data values at varying time lags. If random, such autocorrelations should be near zero for any and all time-lag separations. If non-random, then one or more of the autocorrelations will be significantly non-zero.

In addition, autocorrelation plots are used in the model identification stage for Box-Jenkins autoregressive, moving average time series models.

Sample Plot: Autocorrelations should be near-zero for randomness. Such is not the case in this example and thus the randomness assumption fails



This sample autocorrelation plot shows that the time series is not random, but rather has a high degree of autocorrelation between adjacent and near-adjacent observations.

Definition: r(h) versus h

Autocorrelation plots are formed by

- Vertical axis: Autocorrelation coefficient

  $$R_h = C_h / C_0$$

  where C_h is the autocovariance function

$$C_h = \frac{1}{N} \sum_{t=1}^{N-h} (Y_t - \bar{Y})(Y_{t+h} - \bar{Y})$$

and C_0 is the variance function

$$C_0 = \frac{\sum_{t=1}^{N} (Y_t - \bar{Y})^2}{N}$$

Note--R_h is between -1 and +1.

Note--Some sources may use the following formula for the autocovariance function

$$C_h = \frac{1}{N-h} \sum_{t=1}^{N-h} (Y_t - \bar{Y})(Y_{t+h} - \bar{Y})$$

Although this definition has less bias, the (1/*N*) formulation has some desirable statistical properties and is the form most commonly used in the statistics literature. See [pages 20 and 49-50 in Chatfield] for details.

- Horizontal axis: Time lag *h* (*h* = 1, 2, 3, ...)
- The above line also contains several horizontal reference lines. The middle line is at zero. The other four lines are 95 % and 99 % confidence bands. Note that there are two distinct formulas for generating the confidence bands.

  1. If the autocorrelation plot is being used to test for randomness (i.e., there is no time dependence in the data), the following formula is recommended:

     $$\pm \frac{z_{1-\alpha/2}}{\sqrt{N}}$$

     where *N* is the sample size, *z* is the cumulative distribution function of the standard normal distribution and α is the significance level. In this case, the confidence bands have fixed width that depends on the sample size. This is the formula that was used to generate the confidence bands in the above plot.

  2. Autocorrelation plots are also used in the model identification stage for fitting [ARIMA models]. In this case, a moving average model is assumed for the data and the following confidence bands should be generated:

$$\pm z_{1-\alpha/2}\sqrt{\frac{1}{N}(1+2\sum_{i=1}^{k} y_i^2)}$$

where k is the lag, N is the sample size, z is the cumulative distribution function of the standard normal distribution and α is the significance level. In this case, the confidence bands increase as the lag increases.

Questions

The autocorrelation plot can provide answers to the following questions:

1. Are the data random?
2. Is an observation related to an adjacent observation?
3. Is an observation related to an observation twice-removed? (etc.)
4. Is the observed time series white noise?
5. Is the observed time series sinusoidal?
6. Is the observed time series autoregressive?
7. What is an appropriate model for the observed time series?
8. Is the model

   Y = constant + error

   valid and sufficient?
9. Is the formula $s_{\bar{Y}} = s/\sqrt{N}$ valid?

Importance: Ensure validity of engineering conclusions

Randomness (along with fixed model, fixed variation, and fixed distribution) is one of the four assumptions that typically underlie all measurement processes. The randomness assumption is critically important for the following three reasons:

1. Most standard statistical tests depend on randomness. The validity of the test conclusions is directly linked to the validity of the randomness assumption.

2. Many commonly-used statistical formulae depend on the randomness assumption, the most common formula being the formula for determining the standard deviation of the sample mean:

   $$s_{\bar{Y}} = s/\sqrt{N}$$

   where s is the standard deviation of the data. Although heavily used, the results from using this formula are of no value unless the randomness assumption holds.

3. For univariate data, the default model is

   $$Y = \text{constant} + \text{error}$$

   If the data are not random, this model is incorrect and invalid, and the estimates for the parameters (such as the constant) become nonsensical and invalid.

In short, if the analyst does not check for randomness, then the validity of many of the statistical conclusions becomes suspect. The autocorrelation plot is an excellent way of checking for such randomness.

Examples

Examples of the autocorrelation plot for several common situations are given in the following pages.

1. Random (= White Noise)
2. Weak autocorrelation
3. Strong autocorrelation and autoregressive model
4. Sinusoidal model

Related Techniques

Partial Autocorrelation Plot
Lag Plot
Spectral Plot
Seasonal Subseries Plot

Case Study

The autocorrelation plot is demonstrated in the beam deflection data case study.

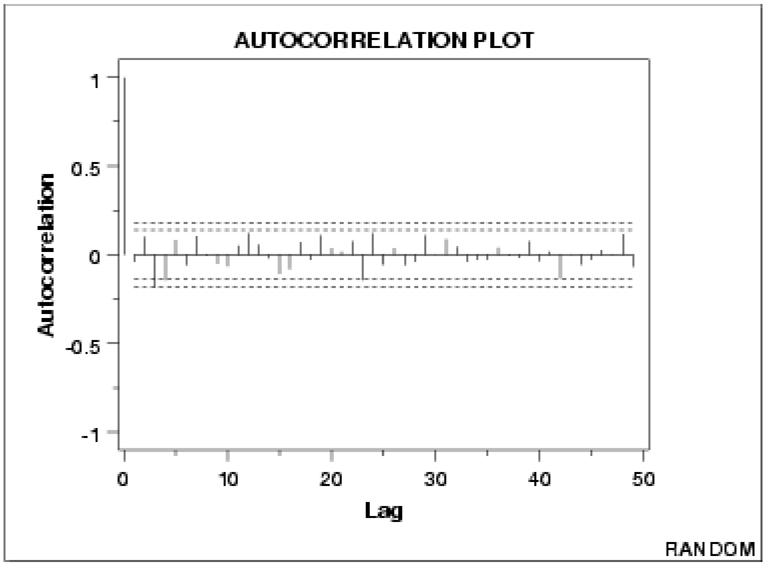
Software

Autocorrelation plots are available in most general purpose statistical software programs.

1.3.3.1.1. Autocorrelation Plot: Random Data

Autocorrelation Plot

The following is a sample autocorrelation plot.



Conclusions

We can make the following conclusions from this plot.

1. There are no significant autocorrelations.
2. The data are random.

Discussion

Note that with the exception of lag 0, which is always 1 by definition, almost all of the autocorrelations fall within the 95% confidence limits. In addition, there is no apparent pattern (such as the first twenty-five being positive and the second twenty-five being negative). This is the abscence of a pattern we expect to see if the data are in fact random.

A few lags slightly outside the 95% and 99% confidence limits do not neccessarily indicate non-randomness. For a 95% confidence interval, we might expect about one out of twenty lags to be statistically significant due to random fluctuations.

There is no associative ability to infer from a current value Y_i as to what the next value Y_{i+1} will be. Such non-association is the essense of randomness. In short, adjacent observations do not "co-relate", so we call this the "no autocorrelation" case.

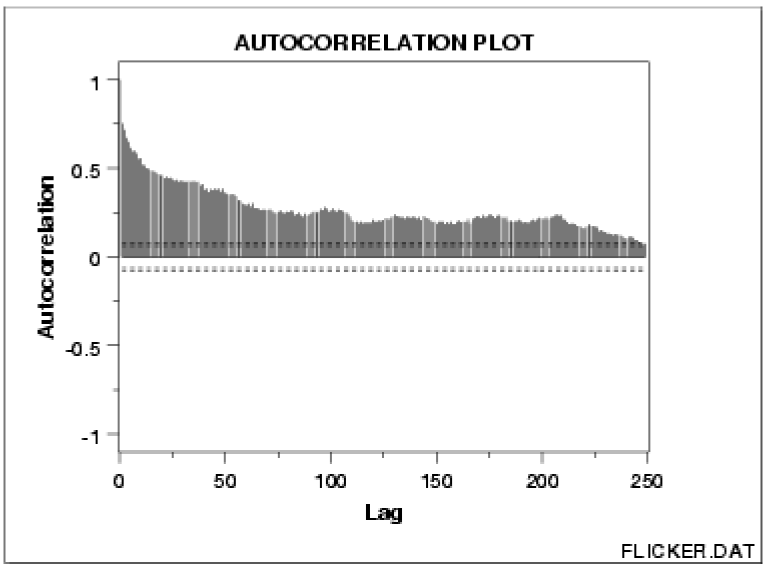
ENGINEERING STATISTICS HANDBOOK
HOME TOOLS & AIDS SEARCH BACK NEXT

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.3. Graphical Techniques: Alphabetic
1.3.3.1. Autocorrelation Plot

1.3.3.1.2. Autocorrelation Plot: Moderate Autocorrelation

Autocorrelation Plot

The following is a sample autocorrelation plot.



Conclusions

We can make the following conclusions from this plot.

1. The data come from an underlying autoregressive model with moderate positive autocorrelation.

Discussion

The plot starts with a moderately high autocorrelation at lag 1 (approximately 0.75) that gradually decreases. The decreasing autocorrelation is generally linear, but with significant noise. Such a pattern is the autocorrelation plot signature of "moderate autocorrelation", which in turn provides moderate predictability if modeled properly.

Recommended Next Step

The next step would be to estimate the parameters for the autoregressive model:

$$Y_i = A_0 + A_1 * Y_{i-1} + E_i$$

Such estimation can be performed by using least squares linear regression or by fitting a Box-Jenkins autoregressive (AR) model.

The randomness assumption for least squares fitting applies to the residuals of the model. That is, even though the original data exhibit non-randomness, the residuals after fitting $\mathbf{Y}_i$ against $\mathbf{Y}_{i-1}$ should result in random residuals. Assessing whether or not the proposed model in fact sufficiently removed the randomness is discussed in detail in the Process Modeling chapter.

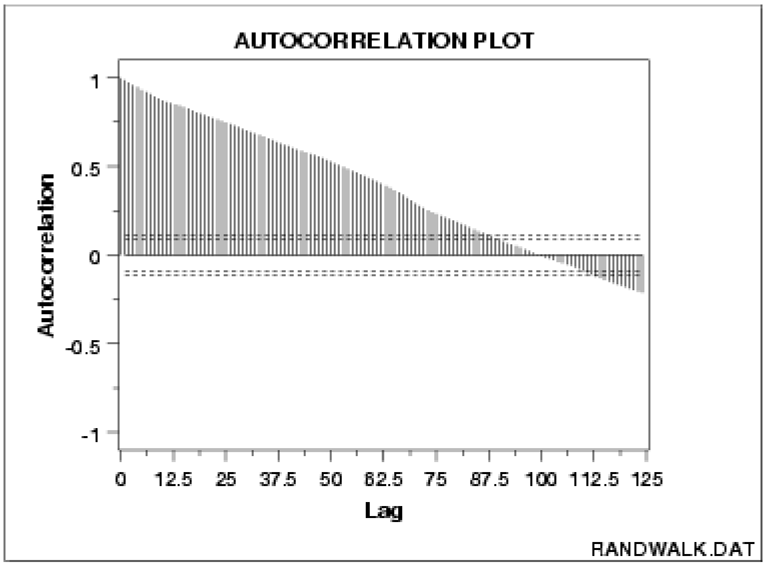
The residual standard deviation for this autoregressive model will be much smaller than the residual standard deviation for the default model

$$Y_i = A_0 + E_i$$

1.3.3.1.3. Autocorrelation Plot: Strong Autocorrelation and Autoregressive Model

Autocorrelation Plot for Strong Autocorrelation

The following is a sample autocorrelation plot.



Conclusions

We can make the following conclusions from the above plot.

1. The data come from an underlying autoregressive model with strong positive autocorrelation.

Discussion

The plot starts with a high autocorrelation at lag 1 (only slightly less than 1) that slowly declines. It continues decreasing until it becomes negative and starts showing an incresing negative autocorrelation. The decreasing autocorrelation is generally linear with little noise. Such a pattern is the autocorrelation plot signature of "strong autocorrelation", which in turn provides high predictability if modeled properly.

Recommended Next Step

The next step would be to estimate the parameters for the autoregressive model:

$$Y_i = A_0 + A_1 * Y_{i-1} + E_i$$

Such estimation can be performed by using least squares linear regression or by fitting a Box-Jenkins autoregressive (AR) model.

The randomness assumption for least squares fitting applies to the residuals of the model. That is, even though the original data exhibit non-randomness, the residuals after fitting $\mathbf{Y}_i$ against $\mathbf{Y}_{i-1}$ should result in random residuals. Assessing whether or not the proposed model in fact sufficiently removed the randomness is discussed in detail in the Process Modeling chapter.

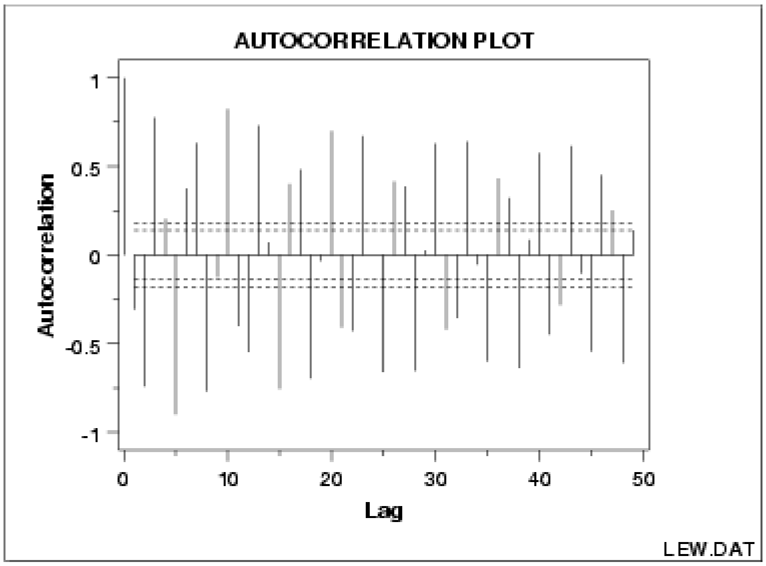
The residual standard deviation for this autoregressive model will be much smaller than the residual standard deviation for the default model

$$Y_i = A_0 + E_i$$

1.3.3.1.4. Autocorrelation Plot: Sinusoidal Model

Autocorrelation Plot for Sinusoidal Model

The following is a sample autocorrelation plot.



Conclusions

We can make the following conclusions from the above plot.

1. The data come from an underlying sinusoidal model.

Discussion

The plot exhibits an alternating sequence of positive and negative spikes. These spikes are not decaying to zero. Such a pattern is the autocorrelation plot signature of a sinusoidal model.

Recommended Next Step

The beam deflection case study gives an example of modeling a sinusoidal model.

1.3.3.2. Bihistogram

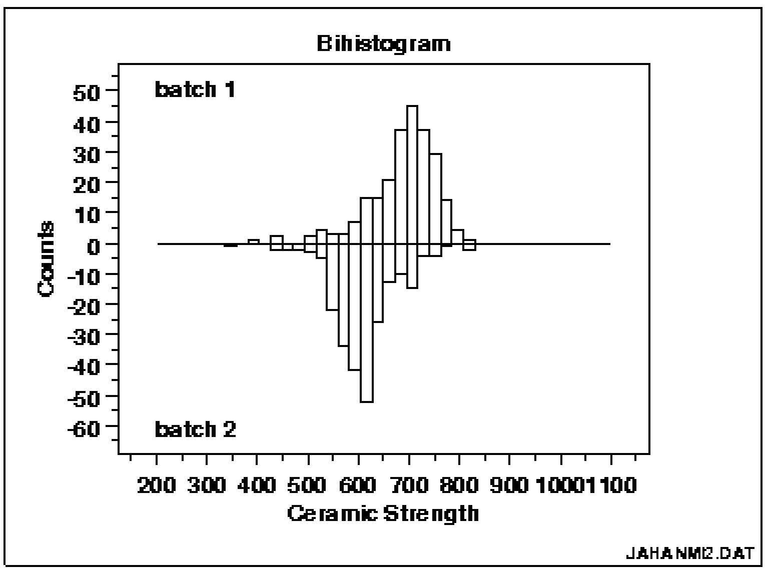
Purpose: Check for a change in location, variation, or distribution

The bihistogram is an EDA tool for assessing whether a before-versus-after engineering modification has caused a change in

- location;
- variation; or
- distribution.

It is a graphical alternative to the two-sample t-test. The bihistogram can be more powerful than the t-test in that all of the distributional features (location, scale, skewness, outliers) are evident on a single plot. It is also based on the common and well-understood histogram.

Sample Plot: This bihistogram reveals that there is a significant difference in ceramic breaking strength between batch 1 (above) and batch 2 (below)



From the above bihistogram, we can see that batch 1 is centered at a ceramic strength value of approximately 725 while batch 2 is centered at a ceramic strength value of approximately 625. That indicates that these batches are displaced by about 100 strength units. Thus the batch factor has a significant effect on the location (typical value) for strength and hence batch is said to be "significant" or to "have an effect". We thus see graphically and convincingly what a t-test or analysis of variance would indicate quantitatively.

With respect to variation, note that the spread (variation) of the above-axis batch 1 histogram does not appear to be that much different from the below-axis batch 2 histogram. With respect to distributional shape, note that the batch 1 histogram is skewed left while the batch 2 histogram is more symmetric with even a hint of a slight skewness to the right.

Thus the bihistogram reveals that there is a clear difference between the batches with respect to location and distribution, but not in regard to variation. Comparing batch 1 and batch 2, we also note that batch 1 is the "better batch" due to its 100-unit higher average strength (around 725).

Definition: Two adjoined histograms

Bihistograms are formed by vertically juxtaposing two histograms:

- Above the axis: Histogram of the response variable for condition 1
- Below the axis: Histogram of the response variable for condition 2

Questions

The bihistogram can provide answers to the following questions:

1. Is a (2-level) factor significant?
2. Does a (2-level) factor have an effect?
3. Does the location change between the 2 subgroups?
4. Does the variation change between the 2 subgroups?
5. Does the distributional shape change between subgroups?
6. Are there any outliers?

Importance: Checks 3 out of the 4 underlying assumptions of a measurement process

The bihistogram is an important EDA tool for determining if a factor "has an effect". Since the bihistogram provides insight into the validity of three (location, variation, and distribution) out of the four (missing only randomness) underlying assumptions in a measurement process, it is an especially valuable tool. Because of the dual (above/below) nature of the plot, the bihistogram is restricted to assessing factors that have only two levels. However, this is very common in the before-versus-after character of many scientific and engineering experiments.

Related Techniques

t test (for shift in location)
F test (for shift in variation)
Kolmogorov-Smirnov test (for shift in distribution)
Quantile-quantile plot (for shift in location and distribution)

Case Study

The bihistogram is demonstrated in the ceramic strength data case study.

Software

The bihistogram is not widely available in general purpose statistical software programs. Bihistograms can be generated using Dataplot and R software.

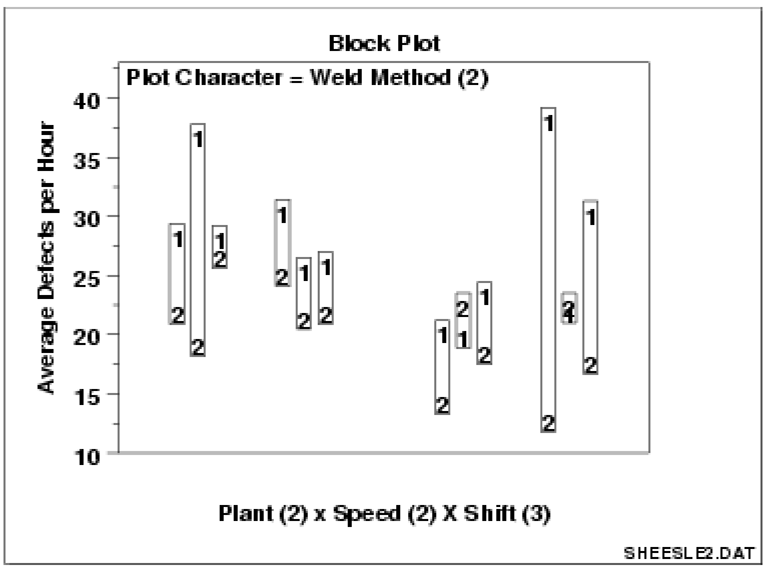
1.3.3.3. Block Plot

Purpose: Check to determine if a factor of interest has an effect robust over all other factors

The block plot (Filliben 1993) is an EDA tool for assessing whether the factor of interest (the primary factor) has a statistically significant effect on the response, and whether that conclusion about the primary factor effect is valid robustly over all other nuisance or secondary factors in the experiment.

It replaces the analysis of variance test with a less assumption-dependent binomial test and should be routinely used whenever we are trying to robustly decide whether a primary factor has an effect.

Sample Plot: Weld method 2 is lower (better) than weld method 1 in 10 of 12 cases



This block plot reveals that in 10 of the 12 cases (bars), weld method 2 is lower (better) than weld method 1. From a binomial point of view, weld method is statistically significant.

Definition

Block Plots are formed as follows:

- Vertical axis: Response variable Y
- Horizontal axis: All combinations of all levels of all nuisance (secondary) factors X1, X2, ...
- Plot Character: Levels of the primary factor XP

Discussion: Primary factor is denoted by plot character: within-bar plot character.

Average number of defective lead wires per hour from a study with four factors,

1. weld strength (2 levels)
2. plant (2 levels)
3. speed (2 levels)
4. shift (3 levels)

are shown in the plot above. Weld strength is the primary factor and the other three factors are nuisance factors. The 12 distinct positions along the horizontal axis correspond to all possible combinations of the three nuisance factors, i.e., 12 = 2 plants x 2 speeds x 3 shifts. These 12 conditions provide the framework for assessing whether any conclusions about the 2 levels of the primary factor (weld method) can truly be called "general conclusions". If we find that one weld method setting does better (smaller average defects per hour) than the other weld method setting for all or most of these 12 nuisance factor combinations, then the conclusion is in fact general and robust.

Ordering along the horizontal axis

In the above chart, the ordering along the horizontal axis is as follows:

- The left 6 bars are from plant 1 and the right 6 bars are from plant 2.
- The first 3 bars are from speed 1, the next 3 bars are from speed 2, the next 3 bars are from speed 1, and the last 3 bars are from speed 2.
- Bars 1, 4, 7, and 10 are from the first shift, bars 2, 5, 8, and 11 are from the second shift, and bars 3, 6, 9, and 12 are from the third shift.

Setting 2 is better than setting 1 in 10 out of 12 cases

In the block plot for the first bar (plant 1, speed 1, shift 1), weld method 1 yields about 28 defects per hour while weld method 2 yields about 22 defects per hour--hence the difference for this combination is about 6 defects per hour and weld method 2 is seen to be better (smaller number of defects per hour).

Is "weld method 2 is better than weld method 1" a general conclusion?

For the second bar (plant 1, speed 1, shift 2), weld method 1 is about 37 while weld method 2 is only about 18. Thus weld method 2 is again seen to be better than weld method 1. Similarly for bar 3 (plant 1, speed 1, shift 3), we see weld method 2 is smaller than weld method 1. Scanning over all of the 12 bars, we see that weld method 2 is smaller than weld method 1 in 10 of the 12 cases, which is highly suggestive of a robust weld method effect.

An event

What is the chance of 10 out of 12 happening by chance?

with chance probability of only 2%

This is probabilistically equivalent to testing whether a coin is fair by flipping it and getting 10 heads in 12 tosses. The chance (from the binomial distribution) of getting 10 (or more extreme: 11, 12) heads in 12 flips of a fair coin is about 2%. Such low-probability events are usually rejected as untenable and in practice we would conclude that there is a difference in weld methods.

Advantage: Graphical and binomial

The advantages of the block plot are as follows:

- A quantitative procedure (analysis of variance) is replaced by a graphical procedure.
- An F-test (analysis of variance) is replaced with a binomial test, which requires fewer assumptions.

Questions

The block plot can provide answers to the following questions:

1. Is the factor of interest significant?
2. Does the factor of interest have an effect?
3. Does the location change between levels of the primary factor?
4. Has the process improved?
5. What is the best setting (= level) of the primary factor?
6. How much of an average improvement can we expect with this best setting of the primary factor?
7. Is there an interaction between the primary factor and one or more nuisance factors?
8. Does the effect of the primary factor change depending on the setting of some nuisance factor?
9. Are there any outliers?

Importance:

Robustly checks the significance of the factor of interest

The block plot is a graphical technique that pointedly focuses on whether or not the primary factor conclusions are in fact robustly general. This question is fundamentally different from the generic multi-factor experiment question where the analyst asks, "What factors are important and what factors are not" (a screening problem)? Global data analysis techniques, such as analysis of variance, can potentially be improved by local, focused data analysis techniques that take advantage of this difference.

Related Techniques

t test (for shift in location for exactly 2 levels)
ANOVA (for shift in location for 2 or more levels)
Bihistogram (for shift in location, variation, and distribution for exactly 2 levels).

Case Study

The block plot is demonstrated in the ceramic strength data case study.

Software

Block plots are not currently available in most general purpose statistical software programs. However they can be generated using Dataplot and, with some programming, R software.

1.3.3.4. Bootstrap Plot

Purpose: Estimate uncertainty

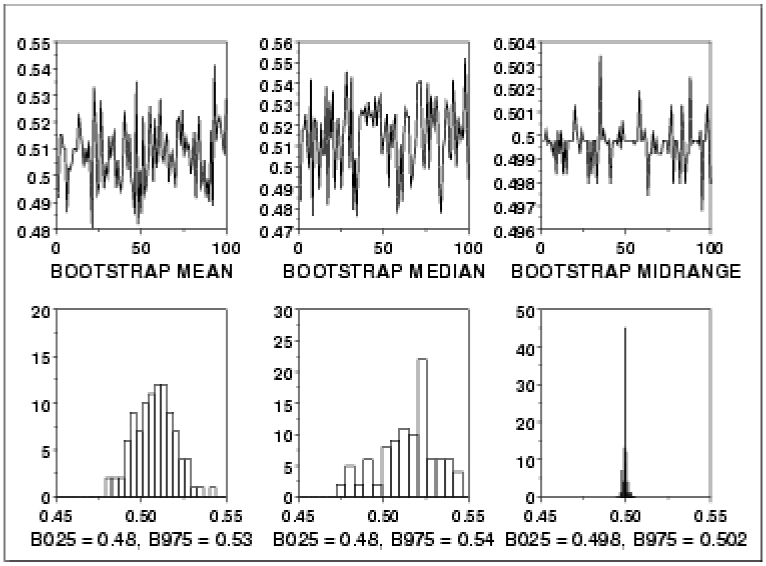
The bootstrap (Efron and Gong) plot is used to estimate the uncertainty of a statistic.

Generate subsamples with replacement

To generate a bootstrap uncertainty estimate for a given statistic from a set of data, a subsample of a size less than or equal to the size of the data set is generated from the data, and the statistic is calculated. This subsample is generated *with replacement* so that any data point can be sampled multiple times or not sampled at all. This process is repeated for many subsamples, typically between 500 and 1000. The computed values for the statistic form an estimate of the sampling distribution of the statistic.

For example, to estimate the uncertainty of the median from a dataset with 50 elements, we generate a subsample of 50 elements and calculate the median. This is repeated at least 500 times so that we have at least 500 values for the median. Although the number of bootstrap samples to use is somewhat arbitrary, 500 subsamples is usually sufficient. To calculate a 90% confidence interval for the median, the sample medians are sorted into ascending order and the value of the 25th median (assuming exactly 500 subsamples were taken) is the lower confidence limit while the value of the 475th median (assuming exactly 500 subsamples were taken) is the upper confidence limit.

Sample Plot:



This bootstrap plot was generated from 500 uniform random numbers. Bootstrap plots and corresponding histograms were generated for the mean, median, and mid-range. The histograms for the corresponding statistics clearly show that for uniform random numbers the mid-range has the smallest variance and is, therefore, a superior location estimator to the mean or the median.

Definition

The bootstrap plot is formed by:

- Vertical axis: Computed value of the desired statistic for a given subsample.
- Horizontal axis: Subsample number.

The bootstrap plot is simply the computed value of the statistic versus the subsample number. That is, the bootstrap plot generates the values for the desired statistic. This is usually immediately followed by a histogram or some other distributional plot to show the location and variation of the sampling distribution of the statistic.

Questions

The bootstrap plot is used to answer the following questions:

- What does the sampling distribution for the statistic look like?
- What is a 95% confidence interval for the statistic?
- Which statistic has a sampling distribution with the smallest variance? That is, which statistic generates the narrowest confidence interval?

Importance

The most common uncertainty calculation is generating a confidence interval for the mean. In this case, the uncertainty formula can be derived mathematically. However, there are many situations in which the uncertainty formulas are mathematically intractable. The bootstrap provides a method for calculating the uncertainty in these cases.

Cautuion on use of the bootstrap	The bootstrap is not appropriate for all distributions and statistics (Efron and Tibrashani). For example, because of the shape of the uniform distribution, the bootstrap is not appropriate for estimating the distribution of statistics that are heavily dependent on the tails, such as the range.
Related Techniques	Histogram Jackknife The jacknife is a technique that is closely related to the bootstrap. The jackknife is beyond the scope of this handbook. See the Efron and Gong article for a discussion of the jackknife.
Case Study	The bootstrap plot is demonstrated in the uniform random numbers case study.
Software	The bootstrap is becoming more common in general purpose statistical software programs. However, it is still not supported in many of these programs. Both R software and Dataplot support a bootstrap capability.

1.3.3.5. Box-Cox Linearity Plot

Purpose: Find the transformation of the X variable that maximizes the correlation between a Y and an X variable

When performing a linear fit of Y against X, an appropriate transformation of X can often significantly improve the fit. The Box-Cox transformation (Box and Cox, 1964) is a particularly useful family of transformations. It is defined as:

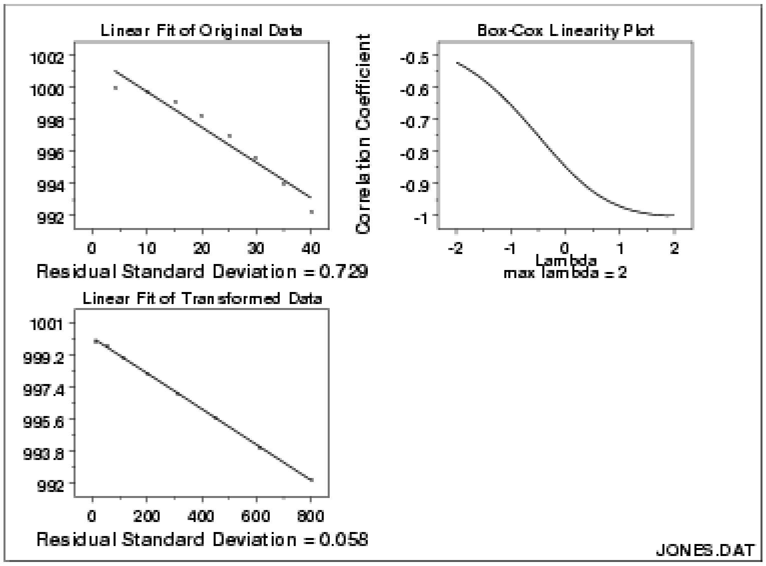
$$T(X) = (X^{\lambda} - 1)/\lambda$$

where X is the variable being transformed and λ is the transformation parameter. For $\lambda = 0$, the natural log of the data is taken instead of using the above formula.

The Box-Cox linearity plot is a plot of the correlation between Y and the transformed X for given values of λ. That is, λ is the coordinate for the horizontal axis variable and the value of the correlation between Y and the transformed X is the coordinate for the vertical axis of the plot. The value of λ corresponding to the maximum correlation (or minimum for negative correlation) on the plot is then the optimal choice for λ.

Transforming X is used to improve the fit. The Box-Cox transformation applied to Y can be used as the basis for meeting the error assumptions. That case is not covered here. See page 225 of (Draper and Smith, 1981) or page 77 of (Ryan, 1997) for a discussion of this case.

Sample Plot



The plot of the original data with the predicted values from a linear fit indicate that a quadratic fit might be preferable. The Box-Cox linearity plot shows a value of $\lambda = 2.0$. The plot of the transformed data with the predicted values from a linear fit with the transformed data shows a better fit (verified by the significant reduction in the residual standard deviation).

Definition

Box-Cox linearity plots are formed by

- Vertical axis: Correlation coefficient from the transformed X and Y
- Horizontal axis: Value for λ

Questions

The Box-Cox linearity plot can provide answers to the following questions:

1. Would a suitable transformation improve my fit?
2. What is the optimal value of the transformation parameter?

Importance: Find a suitable transformation

Transformations can often significantly improve a fit. The Box-Cox linearity plot provides a convenient way to find a suitable transformation without engaging in a lot of trial and error fitting.

Related Techniques

Linear Regression
Box-Cox Normality Plot

Case Study

The Box-Cox linearity plot is demonstrated in the Alaska pipeline data case study.

Software

Box-Cox linearity plots are not a standard part of most general purpose statistical software programs. However, the underlying technique is based on a transformation and computing a correlation coefficient. So if a statistical program supports these capabilities, writing a macro for a Box-Cox linearity plot should be feasible.

1.3.3.6. Box-Cox Normality Plot

Purpose: Find transformation to normalize data

Many statistical tests and intervals are based on the assumption of normality. The assumption of normality often leads to tests that are simple, mathematically tractable, and powerful compared to tests that do not make the normality assumption. Unfortunately, many real data sets are in fact not approximately normal. However, an appropriate transformation of a data set can often yield a data set that does follow approximately a normal distribution. This increases the applicability and usefulness of statistical techniques based on the normality assumption.

The Box-Cox transformation is a particulary useful family of transformations. It is defined as:

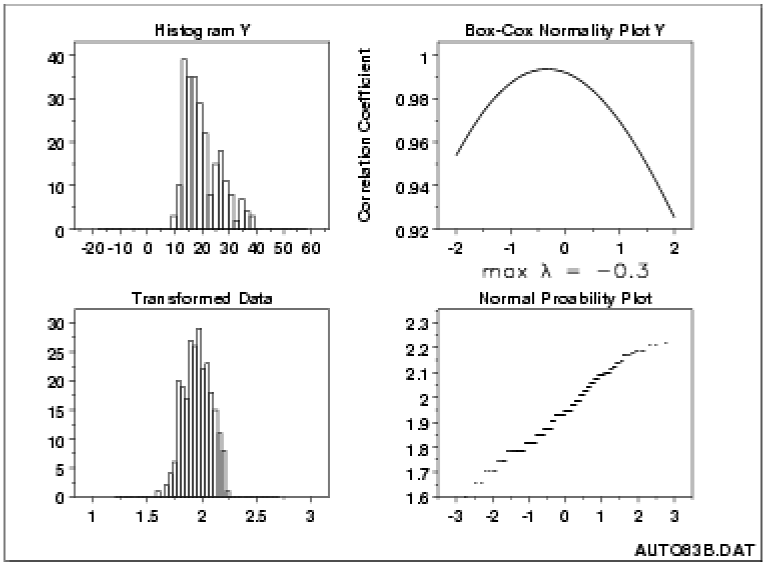
$$T(Y) = (Y^{\lambda} - 1)/\lambda$$

where Y is the response variable and λ is the transformation parameter. For λ = 0, the natural log of the data is taken instead of using the above formula.

Given a particular transformation such as the Box-Cox transformation defined above, it is helpful to define a measure of the normality of the resulting transformation. One measure is to compute the correlation coefficient of a normal probability plot. The correlation is computed between the vertical and horizontal axis variables of the probability plot and is a convenient measure of the linearity of the probability plot (the more linear the probability plot, the better a normal distribution fits the data).

The Box-Cox normality plot is a plot of these correlation coefficients for various values of the λ parameter. The value of λ corresponding to the maximum correlation on the plot is then the optimal choice for λ.

Sample Plot



The histogram in the upper left-hand corner shows a data set that has significant right skewness (and so does not follow a normal distribution). The Box-Cox normality plot shows that the maximum value of the correlation coefficient is at λ = -0.3. The histogram of the data after applying the Box-Cox transformation with λ = -0.3 shows a data set for which the normality assumption is reasonable. This is verified with a normal probability plot of the transformed data.

Definition

Box-Cox normality plots are formed by:

- Vertical axis: Correlation coefficient from the normal probability plot after applying Box-Cox transformation
- Horizontal axis: Value for λ

Questions

The Box-Cox normality plot can provide answers to the following questions:

1. Is there a transformation that will normalize my data?
2. What is the optimal value of the transformation parameter?

Importance: Normalization Improves Validity of Tests

Normality assumptions are critical for many univariate intervals and hypothesis tests. It is important to test the normality assumption. If the data are in fact clearly not normal, the Box-Cox normality plot can often be used to find a transformation that will approximately normalize the data.

Related Techniques

Normal Probability Plot
Box-Cox Linearity Plot

Software

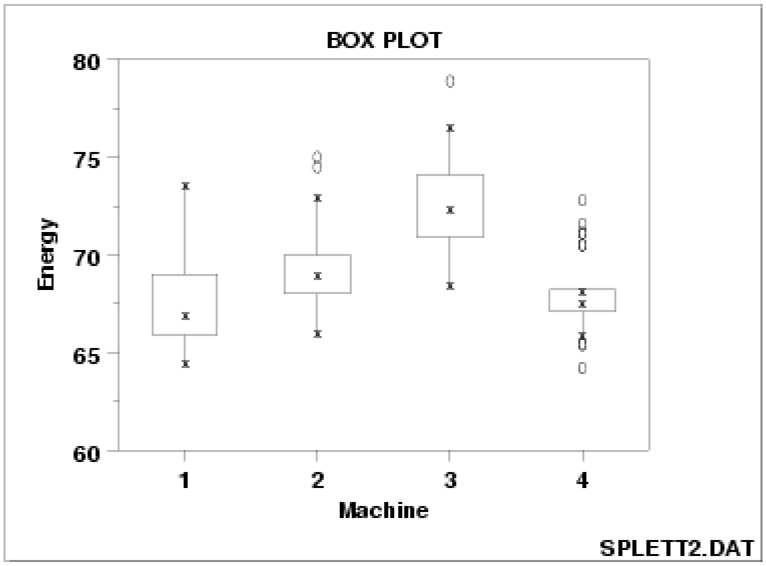
Box-Cox normality plots are not a standard part of most general purpose statistical software programs. However, the underlying technique is based on a normal probability plot and computing a correlation coefficient. So if a statistical program supports these capabilities, writing a macro for a Box-Cox normality plot should be feasible.

1.3.3.7. Box Plot

Purpose: Check location and variation shifts

Box plots (Chambers 1983) are an excellent tool for conveying location and variation information in data sets, particularly for detecting and illustrating location and variation changes between different groups of data.

Sample Plot: This box plot reveals that machine has a significant effect on energy with respect to location and possibly variation



This box plot, comparing four machines for energy output, shows that machine has a significant effect on energy with respect to both location and variation. Machine 3 has the highest energy response (about 72.5); machine 4 has the least variable energy response with about 50% of its readings being within 1 energy unit.

Definition

Box plots are formed by

> Vertical axis: Response variable
> Horizontal axis: The factor of interest

More specifically, we

1. Calculate the median and the quartiles (the lower quartile is the 25th percentile and the upper quartile is the 75th percentile).

2. Plot a symbol at the median (or draw a line) and draw a box (hence the name--box plot) between the lower and upper quartiles; this box represents the middle 50% of the data--the "body" of the data.

3. Draw a line from the lower quartile to the minimum point and another line from the upper quartile to the maximum point. Typically a symbol is drawn at these minimum and maximum points, although this is optional.

Thus the box plot identifies the middle 50% of the data, the median, and the extreme points.

Single or multiple box plots can be drawn

A single box plot can be drawn for one batch of data with no distinct groups. Alternatively, multiple box plots can be drawn together to compare multiple data sets or to compare groups in a single data set. For a single box plot, the width of the box is arbitrary. For multiple box plots, the width of the box plot can be set proportional to the number of points in the given group or sample (some software implementations of the box plot simply set all the boxes to the same width).

Box plots with fences

There is a useful variation of the box plot that more specifically identifies outliers. To create this variation:

1. Calculate the median and the lower and upper quartiles.

2. Plot a symbol at the median and draw a box between the lower and upper quartiles.

3. Calculate the interquartile range (the difference between the upper and lower quartile) and call it IQ.

4. Calculate the following points:

   L1 = lower quartile - 1.5*IQ
   L2 = lower quartile - 3.0*IQ
   U1 = upper quartile + 1.5*IQ
   U2 = upper quartile + 3.0*IQ

5. The line from the lower quartile to the minimum is now drawn from the lower quartile to the smallest point that is greater than L1. Likewise, the line from the upper quartile to the maximum is now drawn to the largest point smaller than U1.

6. Points between L1 and L2 or between U1 and U2 are drawn as small circles. Points less than L2 or greater than U2 are drawn as large circles.

Questions

The box plot can provide answers to the following questions:

1. Is a factor significant?
2. Does the location differ between subgroups?
3. Does the variation differ between subgroups?
4. Are there any outliers?

Importance:

Check the significance of a factor

The box plot is an important EDA tool for determining if a factor has a significant effect on the response with respect to either location or variation.

The box plot is also an effective tool for summarizing large quantities of information.

Related Techniques

Mean Plot
Analysis of Variance

Case Study

The box plot is demonstrated in the ceramic strength data case study.

Software

Box plots are available in most general purpose statistical software programs.

1.3.3.8. Complex Demodulation Amplitude Plot

Purpose: Detect Changing Amplitude in Sinusoidal Models

In the frequency analysis of time series models, a common model is the sinusoidal model:

$$Y_i = C + \alpha \sin(2\pi\omega t_i + \phi) + E_i$$

In this equation, α is the amplitude, ϕ is the phase shift, and ω is the dominant frequency. In the above model, α and ϕ are constant, that is they do not vary with time, t_i.

The complex demodulation amplitude plot (Granger, 1964) is used to determine if the assumption of constant amplitude is justifiable. If the slope of the complex demodulation amplitude plot is not zero, then the above model is typically replaced with the model:

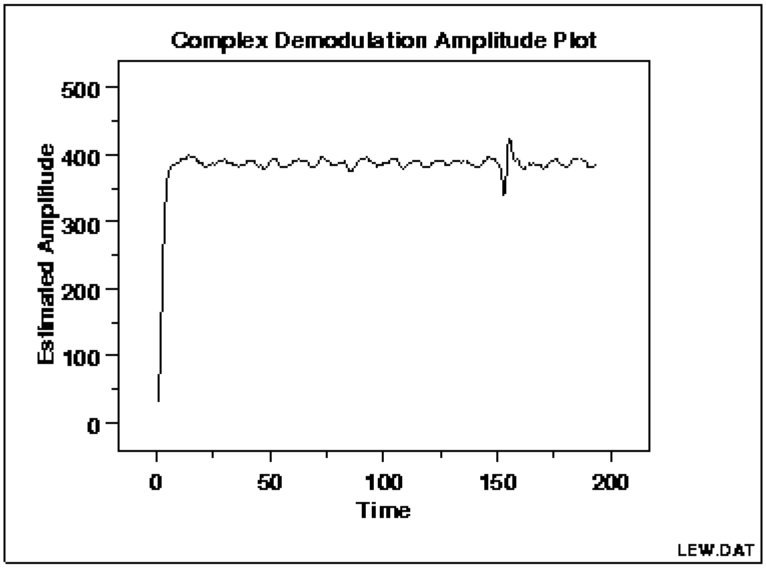
$$Y_i = C + \alpha_i \sin(2\pi\omega t_i + \phi) + E_i$$

where $\hat{\alpha}_i$ is some type of linear model fit with standard least squares. The most common case is a linear fit, that is the model becomes

$$Y_i = C + (B_0 + B_1 * t_i) \sin(2\pi\omega t_i + \phi) + E_i$$

Quadratic models are sometimes used. Higher order models are relatively rare.

Sample Plot:



This complex demodulation amplitude plot shows that:

- the amplitude is fixed at approximately 390;
- there is a start-up effect; and
- there is a change in amplitude at around x = 160 that should be investigated for an outlier.

Definition:

The complex demodulation amplitude plot is formed by:

- Vertical axis: Amplitude
- Horizontal axis: Time

The mathematical computations for determining the amplitude are beyond the scope of the Handbook. Consult Granger (Granger, 1964) for details.

Questions

The complex demodulation amplitude plot answers the following questions:

1. Does the amplitude change over time?
2. Are there any outliers that need to be investigated?
3. Is the amplitude different at the beginning of the series (i.e., is there a start-up effect)?

Importance: Assumption Checking

As stated previously, in the frequency analysis of time series models, a common model is the sinusoidal model:

$$Y_i = C + \alpha \sin\left(2\pi\omega t_i + \phi\right) + E_i$$

In this equation, α is assumed to be constant, that is it does not vary with time. It is important to check whether or not this assumption is reasonable.

The complex demodulation amplitude plot can be used to verify this assumption. If the slope of this plot is essentially zero, then the assumption of constant amplitude is justified. If it is not, α should be replaced with some type of time-varying model. The most common cases are linear ($B_0 + B_1*t$) and quadratic ($B_0 + B_1*t + B_2*t^2$).

Related Techniques

Spectral Plot
Complex Demodulation Phase Plot
Non-Linear Fitting

Case Study

The complex demodulation amplitude plot is demonstrated in the beam deflection data case study.

Software

Complex demodulation amplitude plots are available in some, but not most, general purpose statistical software programs.

1.3.3.9. Complex Demodulation Phase Plot

Purpose: Improve the estimate of frequency in sinusoidal time series models

As stated previously, in the frequency analysis of time series models, a common model is the sinusoidal model:

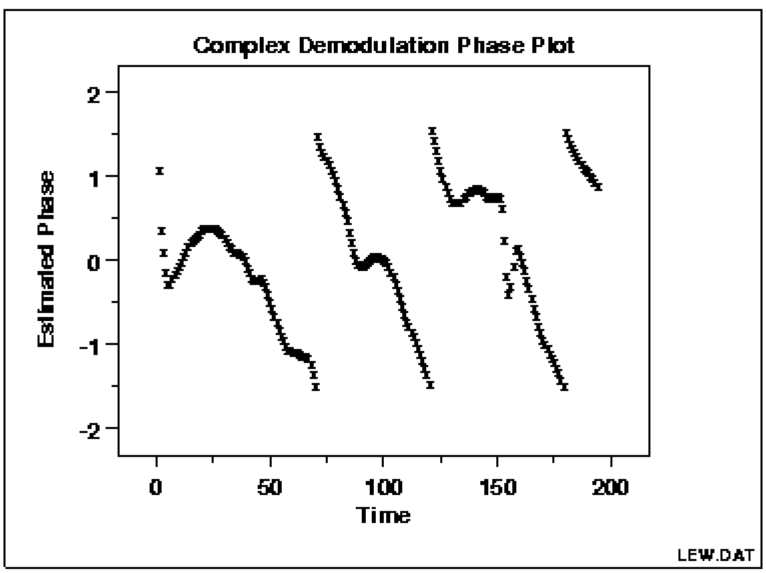
$$Y_i = C + \alpha \sin(2\pi\omega t_i + \phi) + E_i$$

In this equation, α is the amplitude, ϕ is the phase shift, and ω is the dominant frequency. In the above model, α and ϕ are constant, that is they do not vary with time t_i.

The complex demodulation phase plot (Granger, 1964) is used to improve the estimate of the frequency (i.e., ω) in this model.

If the complex demodulation phase plot shows lines sloping from left to right, then the estimate of the frequency should be increased. If it shows lines sloping right to left, then the frequency should be decreased. If there is essentially zero slope, then the frequency estimate does not need to be modified.

Sample Plot:



This complex demodulation phase plot shows that:

the specified demodulation frequency is incorrect;
- the demodulation frequency should be increased.

Definition

The complex demodulation phase plot is formed by:

- Vertical axis: Phase
- Horizontal axis: Time

The mathematical computations for the phase plot are beyond the scope of the Handbook. Consult Granger (Granger, 1964) for details.

Questions

The complex demodulation phase plot answers the following question:

Is the specified demodulation frequency correct?

Importance of a Good Initial Estimate for the Frequency

The non-linear fitting for the sinusoidal model:

$$Y_i = C + \alpha \sin\left(2\pi\omega t_i + \phi\right) + E_i$$

is usually quite sensitive to the choice of good starting values. The initial estimate of the frequency, ω, is obtained from a spectral plot. The complex demodulation phase plot is used to assess whether this estimate is adequate, and if it is not, whether it should be increased or decreased. Using the complex demodulation phase plot with the spectral plot can significantly improve the quality of the non-linear fits obtained.

Related Techniques

Spectral Plot
Complex Demodulation Phase Plot
Non-Linear Fitting

Case Study

The complex demodulation amplitude plot is demonstrated in the beam deflection data case study.

Software

Complex demodulation phase plots are available in some, but not most, general purpose statistical software programs.

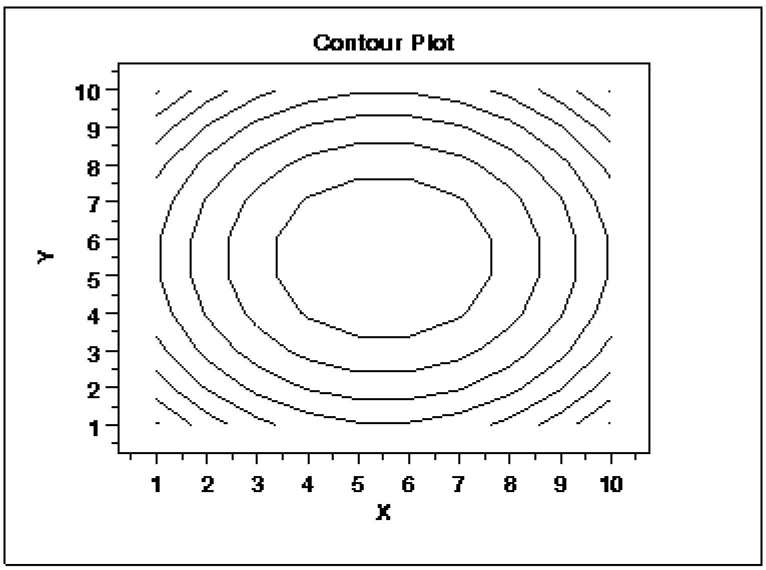
1.3.3.10. Contour Plot

Purpose: Display 3-d surface on 2-d plot

A contour plot is a graphical technique for representing a 3-dimensional surface by plotting constant *z* slices, called contours, on a 2-dimensional format. That is, given a value for *z*, lines are drawn for connecting the (x,y) coordinates where that *z* value occurs.

The contour plot is an alternative to a 3-D surface plot.

Sample Plot:



This contour plot shows that the surface is symmetric and peaks in the center.

Definition

The contour plot is formed by:

- Vertical axis: Independent variable 2
- Horizontal axis: Independent variable 1
- Lines: iso-response values

The independent variables are usually restricted to a regular grid. The actual techniques for determining the correct iso-response values are rather complex and are almost always computer generated.

An additional variable may be required to specify the Z values for drawing the iso-lines. Some software packages require explicit values. Other software packages will determine them automatically.

If the data (or function) do not form a regular grid, you typically need to perform a 2-D interpolation to form a regular grid.

Questions

The contour plot is used to answer the question

How does Z change as a function of X and Y?

Importance: Visualizing 3-dimensional data

For univariate data, a run sequence plot and a histogram are considered necessary first steps in understanding the data. For 2-dimensional data, a scatter plot is a necessary first step in understanding the data.

In a similar manner, 3-dimensional data should be plotted. Small data sets, such as result from designed experiments, can typically be represented by block plots, DOE mean plots, and the like ("DOE" stands for "Design of Experiments"). For large data sets, a contour plot or a 3-D surface plot should be considered a necessary first step in understanding the data.

DOE Contour Plot

The DOE contour plot is a specialized contour plot used in the design of experiments. In particular, it is useful for full and fractional designs.

Related Techniques

3-D Plot

Software

Contour plots are available in most general purpose statistical software programs. They are also available in many general purpose graphics and mathematics programs. These programs vary widely in the capabilities for the contour plots they generate. Many provide just a basic contour plot over a rectangular grid while others permit color filled or shaded contours.

Most statistical software programs that support design of experiments will provide a DOE contour plot capability.

1.3.3.10.1. DOE Contour Plot

DOE Contour Plot: Introduction

The DOE contour plot is a specialized contour plot used in the analysis of full and fractional experimental designs. These designs often have a low level, coded as "-1" or "-", and a high level, coded as "+1" or "+" for each factor. In addition, there can optionally be one or more center points. Center points are at the mid-point between the low and high level for each factor and are coded as "0".

The DOE contour plot is generated for two factors. Typically, this would be the two most important factors as determined by previous analyses (e.g., through the use of the DOE mean plots and an analysis of variance). If more than two factors are important, you may want to generate a series of DOE contour plots, each of which is drawn for two of these factors. You can also generate a matrix of all pairwise DOE contour plots for a number of important factors (similar to the scatter plot matrix for scatter plots).

The typical application of the DOE contour plot is in determining settings that will maximize (or minimize) the response variable. It can also be helpful in determining settings that result in the response variable hitting a pre-determined target value. The DOE contour plot plays a useful role in determining the settings for the next iteration of the experiment. That is, the initial experiment is typically a fractional factorial design with a fairly large number of factors. After the most important factors are determined, the DOE contour plot can be used to help define settings for a full factorial or response surface design based on a smaller number of factors.

Construction of DOE Contour Plot

The following are the primary steps in the construction of the DOE contour plot.

1. The *x* and *y* axes of the plot represent the values of the first and second factor (independent) variables.

2. The four vertex points are drawn. The vertex points are (-1,-1), (-1,1), (1,1), (1,-1). At each vertex point, the average of all the response values at that vertex point is printed.

3. Similarly, if there are center points, a point is drawn at (0,0) and the average of the response values at the center points is printed.

4. The **linear** DOE contour plot assumes the model:

$$Y = \mu + \beta_1 \cdot U_1 + \beta_2 \cdot U_2 + \beta_{12} \cdot U_1 \cdot U_2$$

   where μ is the overall mean of the response variable. The values of β_1, β_2, β_{12}, and μ are estimated from the vertex points using least squares estimation.

   In order to generate a single contour line, we need a value for *Y*, say *Y* . Next, we

0

solve for U_2 in terms of U_1 and, after doing the algebra, we have the equation:

$$U_2 = \frac{(Y_0 - \mu) - \beta_1 \cdot U_1}{\beta_2 + \beta_{12} \cdot U_1}$$

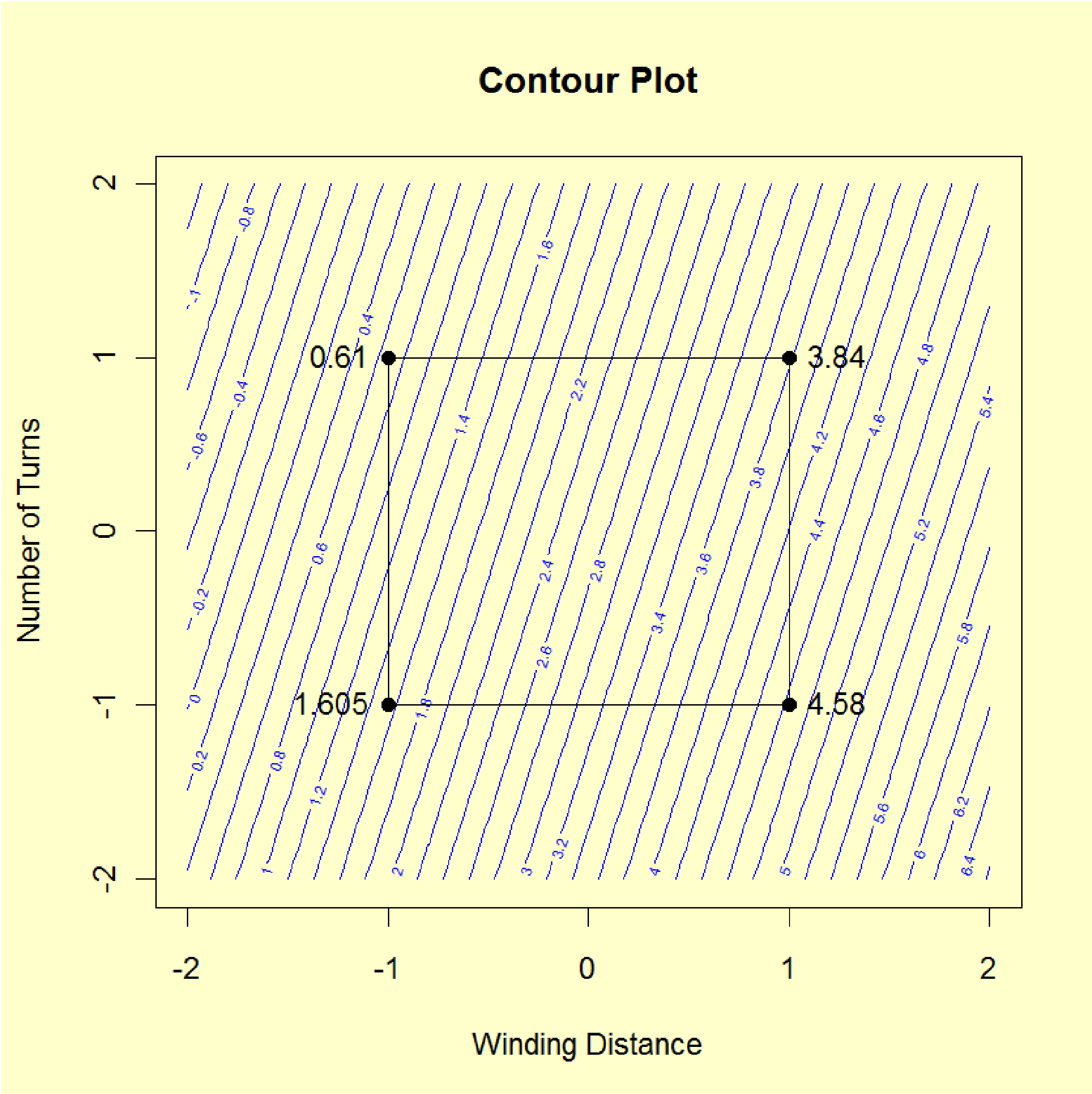
We generate a sequence of points for U_1 in the range -2 to 2 and compute the corresponding values of U_2. These points constitute a single contour line corresponding to $Y = Y_0$.

The user specifies the target values for which contour lines will be generated.

The above algorithm assumes a linear model for the design. DOE contour plots can also be generated for the case in which we assume a quadratic model for the design. The algebra for solving for U_2 in terms of U_1 becomes more complicated, but the fundamental idea is the same. Quadratic models are needed for the case when the average for the center points does not fall in the range defined by the vertex point (i.e., there is curvature).

Sample DOE Contour Plot

The following is a DOE contour plot for the data used in the Eddy current case study. The analysis in that case study demonstrated that X1 and X2 were the most important factors.



Interpretation of the Sample DOE Contour Plot

From the above DOE contour plot we can derive the following information.

1. Interaction significance;
2. Best (data) setting for these two dominant factors;

Interaction Significance

Note the appearance of the contour plot. If the contour curves are linear, then that implies that the interaction term is not significant; if the contour curves have considerable curvature, then that implies that the interaction term is large and important. In our case, the contour curves do not have considerable curvature, and so we conclude that the X1*X2 term is not significant.

Best Settings

To determine the best factor settings for the already-run experiment, we first must define what "best" means. For the Eddy current data set used to generate this DOE contour plot, "best" means to **maximize** (rather than minimize or hit a target) the response. Hence from the contour plot we determine the best settings for the two dominant factors by simply scanning the four vertices and choosing the vertex with the **largest** value (= average response). In this case, it is (X1 = +1, X2 = +1).

As for factor X3, the contour plot provides no best setting information, and so we would resort to other tools: the main effects plot, the interaction effects matrix, or the ordered data to determine optimal X3 settings.

Case Study

The Eddy current case study demonstrates the use of the DOE contour plot in the context of the analysis of a full factorial design.

Software

DOE Contour plots are available in many statistical software programs that analyze data from designed experiments.

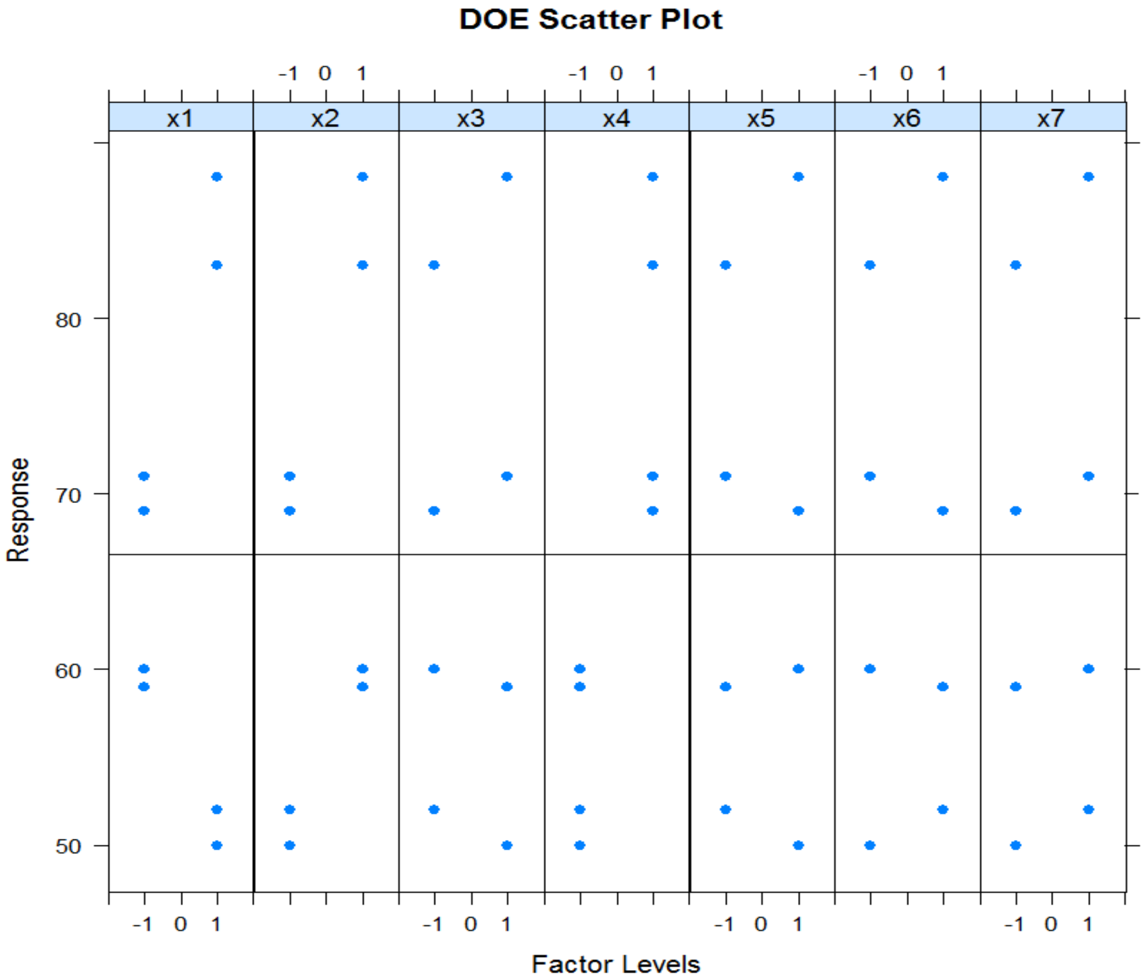
1.3.3.11. DOE Scatter Plot

Purpose: Determine Important Factors with Respect to Location and Scale

The DOE scatter plot shows the response values for each level of each factor (i.e., independent) variable. This graphically shows how the location and scale vary for both within a factor variable and between different factor variables. This graphically shows which are the important factors and can help provide a ranked list of important factors from a designed experiment. The DOE scatter plot is a complement to the traditional analyis of variance of designed experiments.

DOE scatter plots are typically used in conjunction with the DOE mean plot and the DOE standard deviation plot. The DOE mean plot replaces the raw response values with mean response values while the DOE standard deviation plot replaces the raw response values with the standard deviation of the response values. There is value in generating all 3 of these plots. The DOE mean and standard deviation plots are useful in that the summary measures of location and spread stand out (they can sometimes get lost with the raw plot). However, the raw data points can reveal subtleties, such as the presence of outliers, that might get lost with the summary statistics.

Sample Plot: Factors 4, 2, 3, and 7 are the Important Factors.



Description of the Plot

For this sample plot, there are seven factors and each factor has two levels. For each factor, we define a distinct *x* coordinate for each level of the factor. For example, for factor 1, level 1 is coded as 0.8 and level 2 is coded as 1.2. The *y* coordinate is simply the value of the response variable. The solid horizontal line is drawn at the overall mean of the response variable. The vertical dotted lines are added for clarity.

Although the plot can be drawn with an arbitrary number of levels for a factor, it is really only useful when there are two or three levels for a factor.

Conclusions

This sample DOE scatter plot shows that:

1. there does not appear to be any outliers;
2. the levels of factors 2 and 4 show distinct location differences; and
3. the levels of factor 1 show distinct scale differences.

Definition: Response Values Versus Factor Variables

DOE scatter plots are formed by:

- Vertical axis: Value of the response variable
- Horizontal axis: Factor variable (with each level of the factor coded with a slightly offset *x* coordinate)

Questions

The DOE scatter plot can be used to answer the following questions:

1. Which factors are important with respect to location and scale?
2. Are there outliers?

Importance: Identify Important Factors with Respect to Location and Scale

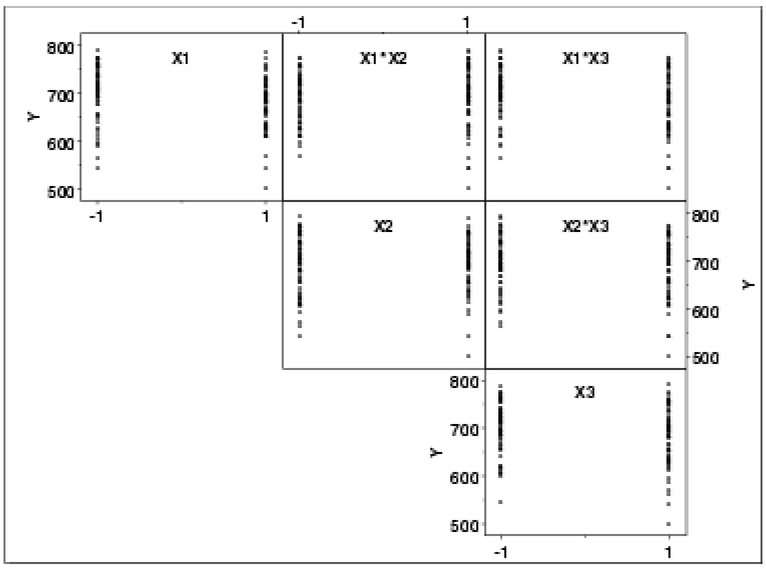
The goal of many designed experiments is to determine which factors are important with respect to location and scale. A ranked list of the important factors is also often of interest. DOE scatter, mean, and standard deviation plots show this graphically. The DOE scatter plot additionally shows if outliers may potentially be distorting the results.

DOE scatter plots were designed primarily for analyzing designed experiments. However, they are useful for any type of multi-factor data (i.e., a response variable with two or more factor variables having a small number of distinct levels) whether or not the data were generated from a designed experiment.

Extension for Interaction Effects

Using the concept of the scatterplot matrix, the DOE scatter plot can be extended to display first order interaction effects.

Specifically, if there are *k* factors, we create a matrix of plots with *k* rows and *k* columns. On the diagonal, the plot is simply a DOE scatter plot with a single factor. For the off-diagonal plots, we multiply the values of X_i and X_j. For the common 2-level designs (i.e., each factor has two levels) the values are typically coded as -1 and 1, so the multiplied values are also -1 and 1. We then generate a DOE scatter plot for this interaction variable. This plot is called a DOE interaction effects plot and an example is shown below.



Interpretation of the DOE Interaction Effects Plot

We can first examine the diagonal elements for the main effects. These diagonal plots show a great deal of overlap between the levels for all three factors. This indicates that location and scale effects will be relatively small.

We can then examine the off-diagonal plots for the first order interaction effects. For example, the plot in the first row and second column is the interaction between factors X1 and X2. As with the main effect plots, no clear patterns are evident.

Related Techniques

DOE mean plot
DOE standard deviation plot
Block plot
Box plot
Analysis of variance

Case Study	The DOE scatter plot is demonstrated in the ceramic strength data case study.
Software	DOE scatter plots are available in some general purpose statistical software programs, although the format may vary somewhat between these programs. They are essentially just scatter plots with the X variable defined in a particular way, so it should be feasible to write macros for DOE scatter plots in most statistical software programs.

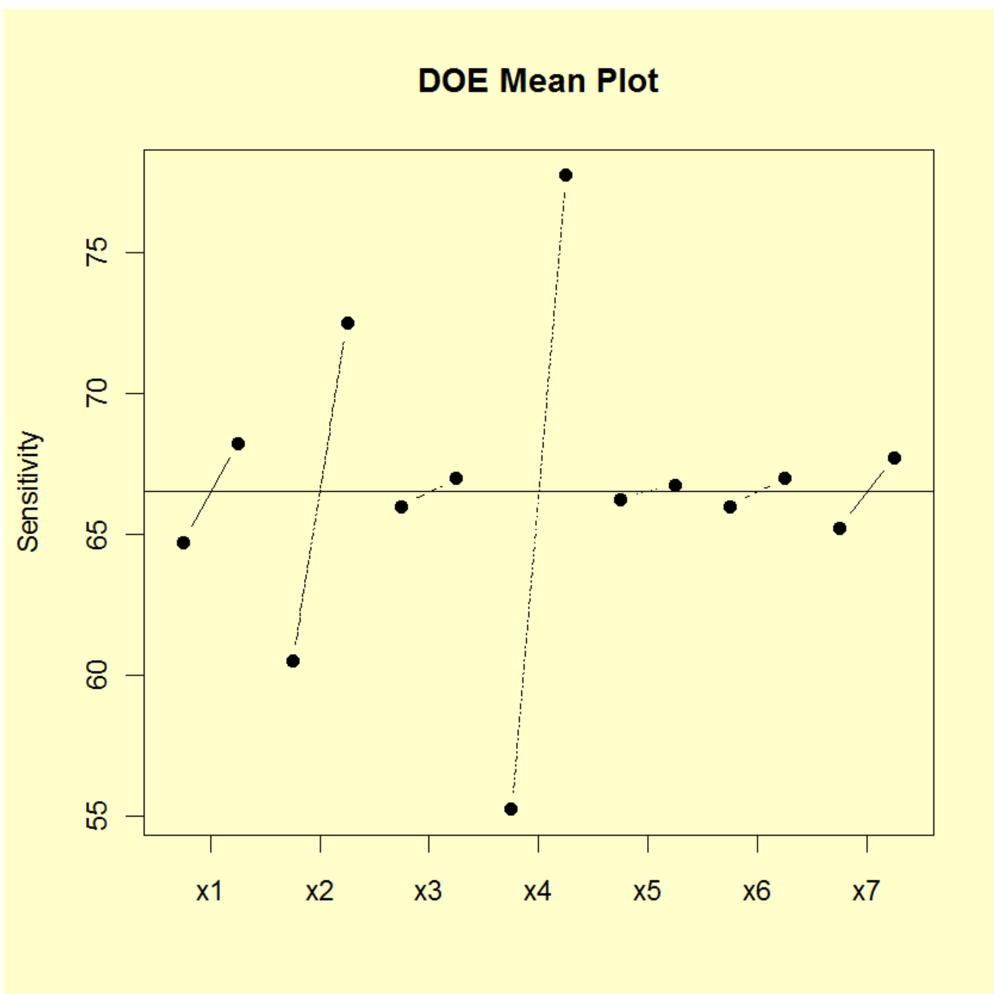
1.3.3.12. DOE Mean Plot

Purpose: Detect Important Factors With Respect to Location

The DOE mean plot is appropriate for analyzing data from a designed experiment, with respect to important factors, where the factors are at two or more levels. The plot shows mean values for the two or more levels of each factor plotted by factor. The means for a single factor are connected by a straight line. The DOE mean plot is a complement to the traditional analysis of variance of designed experiments.

This plot is typically generated for the mean. However, it can be generated for other location statistics such as the median.

Sample Plot: Factors 4, 2, and 1 Are the Most Important Factors



This sample DOE mean plot shows that:

1. factor 4 is the most important;
2. factor 2 is the second most important;
3. factor 1 is the third most important;
4. factor 7 is the fourth most important;
5. factor 6 is the fifth most important;
6. factors 3 and 5 are relatively unimportant.

In summary, factors 4, 2, and 1 seem to be clearly important, factors 3 and 5 seem to be clearly unimportant, and factors 6 and 7 are borderline factors whose inclusion in any subsequent models will be determined by further analyses.

Definition: Mean Response Versus Factor Variables

DOE mean plots are formed by:

- Vertical axis: Mean of the response variable for each level of the factor
- Horizontal axis: Factor variable

Questions

The DOE mean plot can be used to answer the following questions:

1. Which factors are important? The DOE mean plot does not provide a definitive answer to this question, but it does help categorize factors as "clearly important", "clearly not important", and "borderline importance".
2. What is the ranking list of the important factors?

Importance: Determine Significant Factors

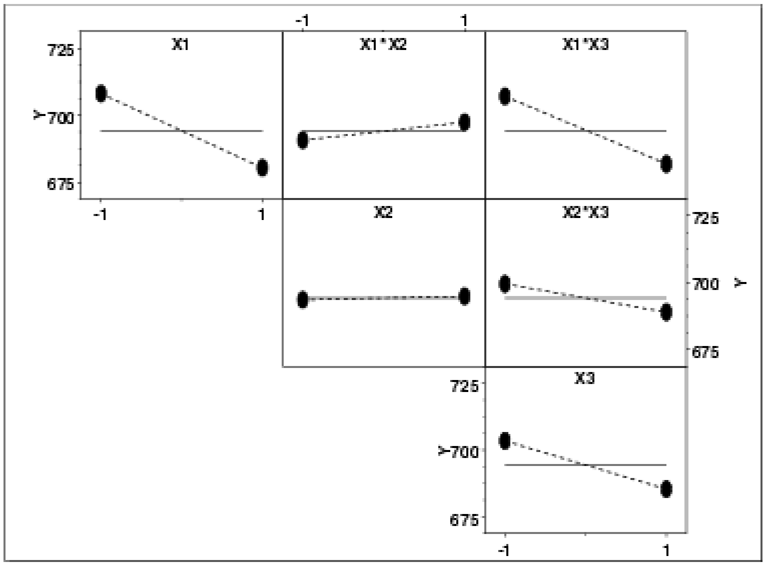
The goal of many designed experiments is to determine which factors are significant. A ranked order listing of the important factors is also often of interest. The DOE mean plot is ideally suited for answering these types of questions and we recommend its routine use in analyzing designed experiments.

Extension for Interaction Effects

Using the concept of the scatter plot matrix, the DOE mean plot can be extended to display first-order interaction effects.

Specifically, if there are k factors, we create a matrix of plots with k rows and k columns. On the diagonal, the plot is simply a DOE mean plot with a single factor. For the off-diagonal plots, measurements at each level of the interaction are plotted versus level, where level is X_i times X_j and X_i is the code for the ith main effect level and X_j is the code for the jth main effect. For the common 2-level designs (i.e., each factor has two levels) the values are typically coded as -1 and 1, so the multiplied values are also -1 and 1. We then generate a DOE mean plot for this interaction variable. This plot is called a DOE interaction effects plot and an example is shown below.

DOE Interaction Effects Plot



This plot shows that the most significant factor is X1 and the most significant interaction is between X1 and X3.

Related Techniques

[DOE scatter plot](#)
[DOE standard deviation plot](#)
[Block plot](#)
[Box plot](#)
[Analysis of variance](#)

Case Study

The DOE mean plot and the DOE interaction effects plot are demonstrated in the [ceramic strength](#) data case study.

Software

DOE mean plots are available in some general purpose statistical software programs, although the format may vary somewhat between these programs. It may be feasible to write macros for DOE mean plots in some statistical software programs that do not support this plot directly.

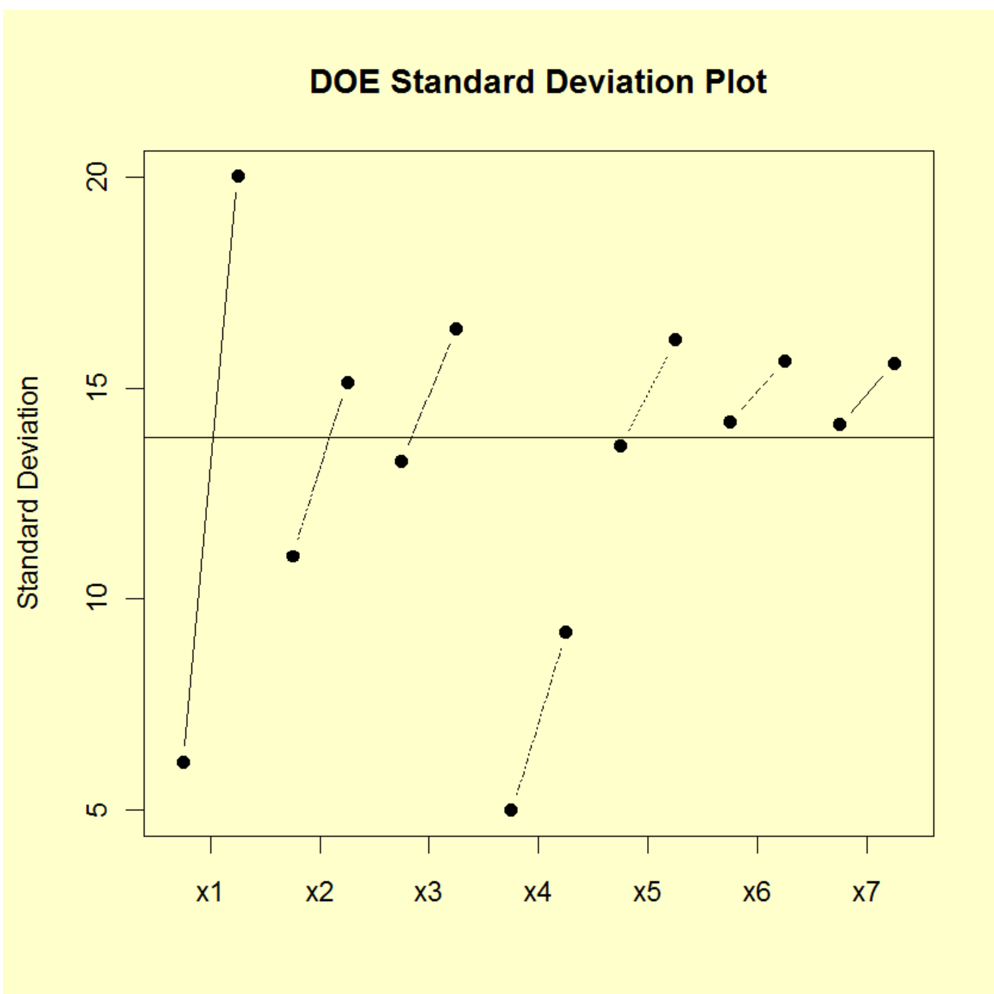
1.3.3.13. DOE Standard Deviation Plot

Purpose: Detect Important Factors With Respect to Scale

The DOE standard deviation plot is appropriate for analyzing data from a designed experiment, with respect to important factors, where the factors are at two or more levels and there are repeated values at each level. The plot shows standard deviation values for the two or more levels of each factor plotted by factor. The standard deviations for a single factor are connected by a straight line. The DOE standard deviation plot is a complement to the traditional analysis of variance of designed experiments.

This plot is typically generated for the standard deviation. However, it can also be generated for other scale statistics such as the range, the median absolute deviation, or the average absolute deviation.

Sample Plot



This sample DOE standard deviation plot shows that:

1. factor 1 has the greatest difference in standard deviations between factor levels;
2. factor 4 has a significantly lower average standard deviation than the average standard deviations of other factors (but the level 1 standard deviation for factor 1 is about the same as the level 1 standard deviation for factor 4);
3. for all factors, the level 1 standard deviation is smaller than the level 2 standard deviation.

Definition: Response Standard Deviations Versus Factor Variables

DOE standard deviation plots are formed by:

- Vertical axis: Standard deviation of the response variable for each level of the factor
- Horizontal axis: Factor variable

Questions

The DOE standard deviation plot can be used to answer the following questions:

1. How do the standard deviations vary across factors?
2. How do the standard deviations vary within a factor?
3. Which are the most important factors with respect to scale?
4. What is the ranked list of the important factors with respect to scale?

Importance: Assess Variability

The goal with many designed experiments is to determine which factors are significant. This is usually determined from the means of the factor levels (which can be conveniently shown with a DOE mean plot). A secondary goal is to assess the variability of the responses both within a factor and between factors. The DOE standard deviation plot is a convenient way to do this.

Related Techniques

DOE scatter plot
DOE mean plot
Block plot
Box plot
Analysis of variance

Case Study

The DOE standard deviation plot is demonstrated in the ceramic strength data case study.

Software

DOE standard deviation plots are not available in most general purpose statistical software programs. It may be feasible to write macros for DOE standard deviation plots in some statistical software programs that do not support them directly.

1.3.3.14. Histogram

Purpose: Summarize a Univariate Data Set

The purpose of a histogram (Chambers) is to graphically summarize the distribution of a univariate data set.

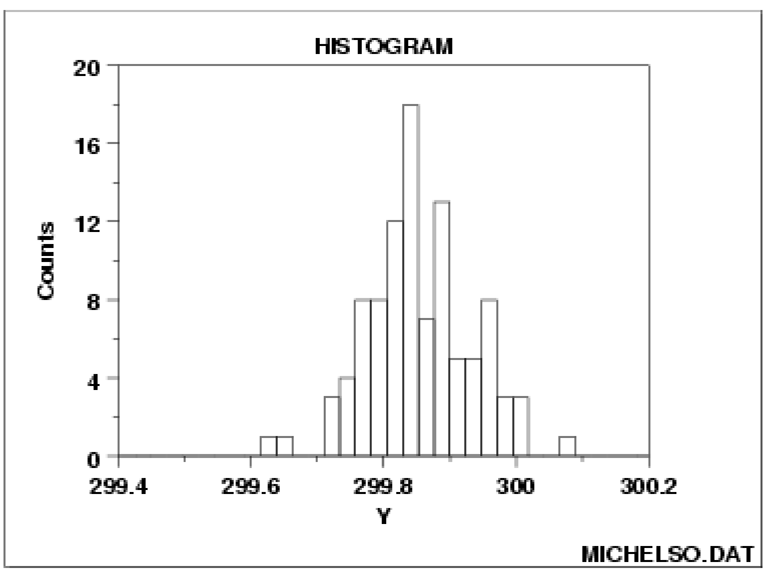
The histogram graphically shows the following:

1. center (i.e., the location) of the data;
2. spread (i.e., the scale) of the data;
3. skewness of the data;
4. presence of outliers; and
5. presence of multiple modes in the data.

These features provide strong indications of the proper distributional model for the data. The probability plot or a goodness-of-fit test can be used to verify the distributional model.

The examples section shows the appearance of a number of common features revealed by histograms.

Sample Plot



Definition

The most common form of the histogram is obtained by splitting the range of the data into equal-sized bins (called classes). Then for each bin, the number of points from the data set that fall into each bin are counted. That is

Vertical axis: Frequency (i.e., counts for each bin)
- Horizontal axis: Response variable

The classes can either be defined arbitrarily by the user or via some systematic rule. A number of theoretically derived rules have been proposed by Scott (Scott 1992).

The cumulative histogram is a variation of the histogram in which the vertical axis gives not just the counts for a single bin, but rather gives the counts for that bin plus all bins for smaller values of the response variable.

Both the histogram and cumulative histogram have an additional variant whereby the counts are replaced by the normalized counts. The names for these variants are the relative histogram and the relative cumulative histogram.

There are two common ways to normalize the counts.

1. The normalized count is the count in a class divided by the total number of observations. In this case the relative counts are normalized to sum to one (or 100 if a percentage scale is used). This is the intuitive case where the height of the histogram bar represents the proportion of the data in each class.

2. The normalized count is the count in the class divided by the number of observations times the class width. For this normalization, the area (or integral) under the histogram is equal to one. From a probabilistic point of view, this normalization results in a relative histogram that is most akin to the probability density function and a relative cumulative histogram that is most akin to the cumulative distribution function. If you want to overlay a probability density or cumulative distribution function on top of the histogram, use this normalization. Although this normalization is less intuitive (relative frequencies greater than 1 are quite permissible), it is the appropriate normalization if you are using the histogram to model a probability density function.

Questions

The histogram can be used to answer the following questions:

1. What kind of population distribution do the data come from?
2. Where are the data located?
3. How spread out are the data?
4. Are the data symmetric or skewed?
5. Are there outliers in the data?

Examples

1. Normal
2. Symmetric, Non-Normal, Short-Tailed

3. Symmetric, Non-Normal, Long-Tailed
4. Symmetric and Bimodal
5. Bimodal Mixture of 2 Normals
6. Skewed (Non-Symmetric) Right
7. Skewed (Non-Symmetric) Left
8. Symmetric with Outlier

Related Techniques

Box plot
Probability plot

The techniques below are not discussed in the Handbook. However, they are similar in purpose to the histogram. Additional information on them is contained in the Chambers and Scott references.

Frequency Plot
Stem and Leaf Plot
Density Trace

Case Study

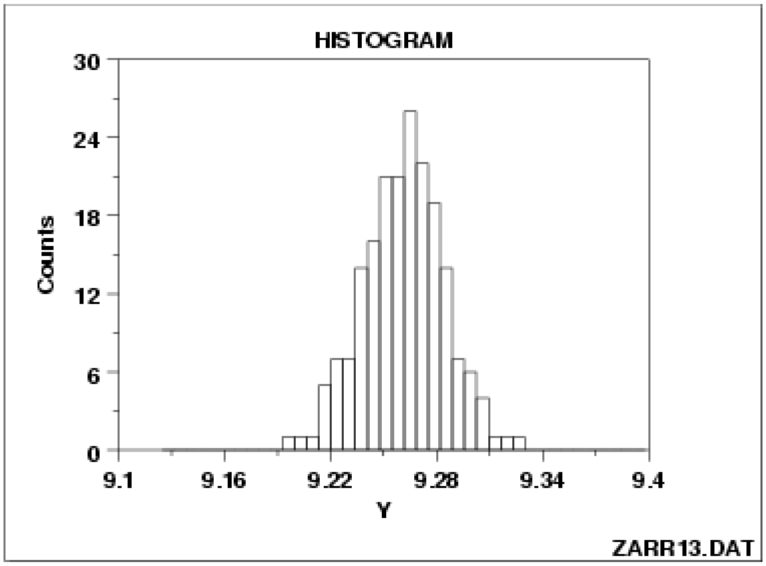
The histogram is demonstrated in the heat flow meter data case study.

Software

Histograms are available in most general purpose statistical software programs. They are also supported in most general purpose charting, spreadsheet, and business graphics programs.

1.3.3.14.1. Histogram Interpretation: Normal

Symmetric, Moderate-Tailed Histogram



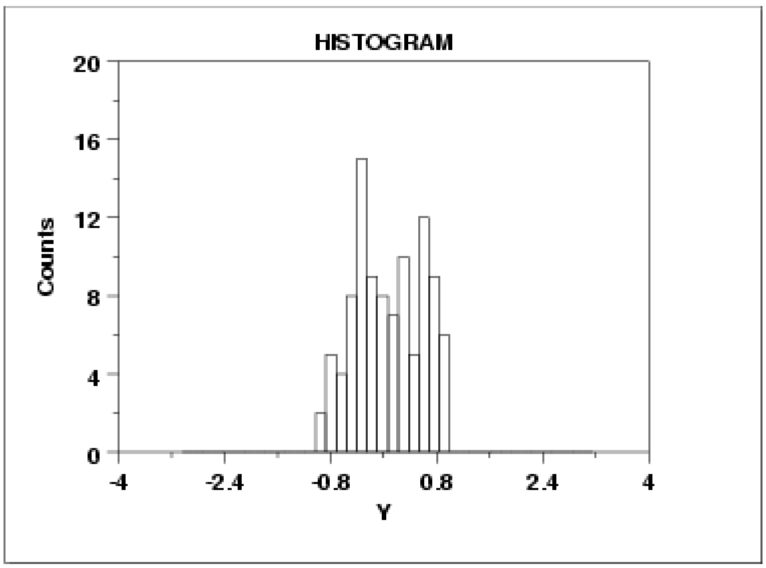
Note the classical bell-shaped, symmetric histogram with most of the frequency counts bunched in the middle and with the counts dying off out in the tails. From a physical science/engineering point of view, the normal distribution is that distribution which occurs most often in nature (due in part to the central limit theorem).

Recommended Next Step

If the histogram indicates a symmetric, moderate tailed distribution, then the recommended next step is to do a normal probability plot to confirm approximate normality. If the normal probability plot is linear, then the normal distribution is a good model for the data.

1.3.3.14.2. Histogram Interpretation: Symmetric, Non-Normal, Short-Tailed

Symmetric, Short-Tailed Histogram



Description of What Short-Tailed Means

For a symmetric distribution, the "body" of a distribution refers to the "center" of the distribution--commonly that region of the distribution where most of the probability resides--the "fat" part of the distribution. The "tail" of a distribution refers to the extreme regions of the distribution--both left and right. The "tail length" of a distribution is a term that indicates how fast these extremes approach zero.

For a short-tailed distribution, the tails approach zero very fast. Such distributions commonly have a truncated ("sawed-off") look. The classical short-tailed distribution is the uniform (rectangular) distribution in which the probability is constant over a given range and then drops to zero everywhere else--we would speak of this as having no tails, or extremely short tails.

For a moderate-tailed distribution, the tails decline to zero in a moderate fashion. The classical moderate-tailed distribution is the normal (Gaussian) distribution.

For a long-tailed distribution, the tails decline to zero very slowly--and hence one is apt to see probability a long way from the body of the distribution. The classical long-tailed distribution is the Cauchy distribution.

In terms of tail length, the histogram shown above would be characteristic of a "short-tailed" distribution.

The optimal (unbiased and most precise) estimator for location for the center of a distribution is heavily dependent on the tail length of the distribution. The common choice of taking N observations and using the calculated sample mean as the best estimate for the center of the distribution is a good choice for the normal distribution (moderate tailed), a poor choice for the uniform distribution (short tailed), and a horrible choice for the Cauchy distribution (long tailed). Although for the normal distribution the sample mean is as precise an estimator as we can get, for the uniform and Cauchy distributions, the sample mean is not the best estimator.

For the uniform distribution, the midrange

midrange = (smallest + largest) / 2

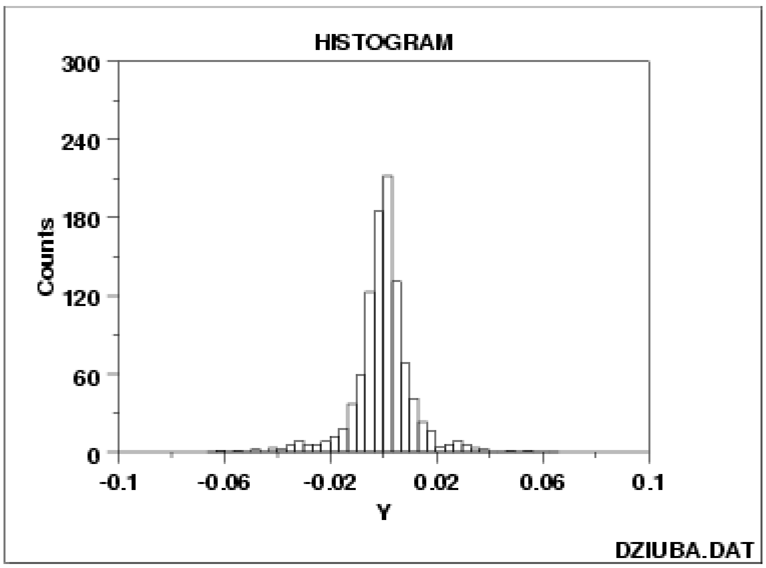
is the best estimator of location. For a Cauchy distribution, the median is the best estimator of location.

Recommended Next Step

If the histogram indicates a symmetric, short-tailed distribution, the recommended next step is to generate a uniform probability plot. If the uniform probability plot is linear, then the uniform distribution is an appropriate model for the data.

1.3.3.14.3. Histogram Interpretation: Symmetric, Non-Normal, Long-Tailed

Symmetric, Long-Tailed Histogram



Description of Long-Tailed

The previous example contains a discussion of the distinction between short-tailed, moderate-tailed, and long-tailed distributions.

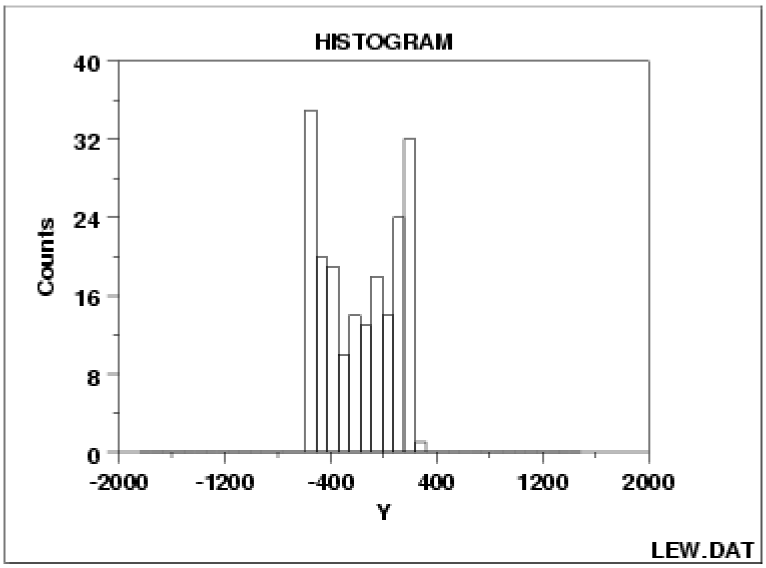
In terms of tail length, the histogram shown above would be characteristic of a "long-tailed" distribution.

Recommended Next Step

If the histogram indicates a symmetric, long tailed distribution, the recommended next step is to do a Cauchy probability plot. If the Cauchy probability plot is linear, then the Cauchy distribution is an appropriate model for the data. Alternatively, a Tukey Lambda PPCC plot may provide insight into a suitable distributional model for the data.

1.3.3.14.4. Histogram Interpretation: Symmetric and Bimodal

Symmetric, Bimodal Histogram



Description of Bimodal

The mode of a distribution is that value which is most frequently occurring or has the largest probability of occurrence. The sample mode occurs at the peak of the histogram.

For many phenomena, it is quite common for the distribution of the response values to cluster around a single mode (unimodal) and then distribute themselves with lesser frequency out into the tails. The normal distribution is the classic example of a unimodal distribution.

The histogram shown above illustrates data from a bimodal (2 peak) distribution. The histogram serves as a tool for diagnosing problems such as bimodality. Questioning the underlying reason for distributional non-unimodality frequently leads to greater insight and improved deterministic modeling of the phenomenon under study. For example, for the data presented above, the bimodal histogram is caused by sinusoidality in the data.

Recommended Next Step

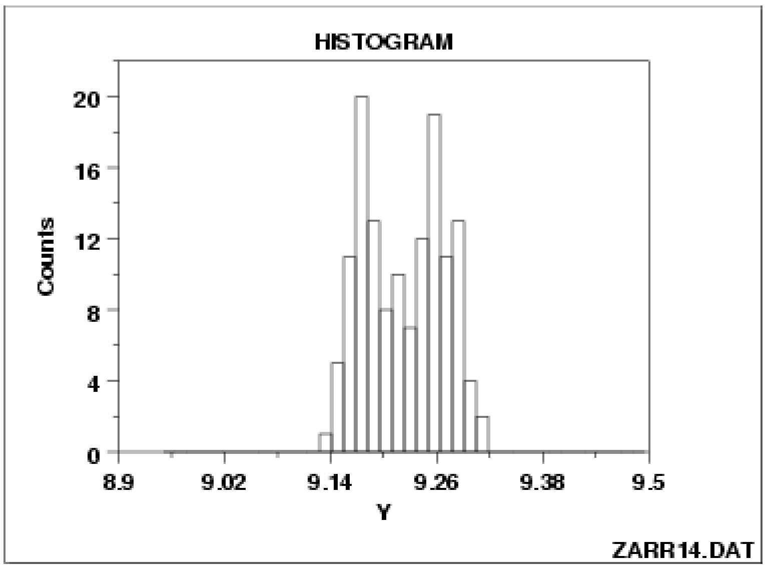
If the histogram indicates a symmetric, bimodal distribution, the recommended next steps are to:

1. Do a run sequence plot or a scatter plot to check for sinusoidality.
2. Do a lag plot to check for sinusoidality. If the lag plot is elliptical, then the data are sinusoidal.
3. If the data are sinusoidal, then a spectral plot is used to graphically estimate the underlying sinusoidal frequency.
4. If the data are not sinusoidal, then a Tukey Lambda PPCC plot may determine the best-fit symmetric distribution for the data.
5. The data may be fit with a mixture of two distributions. A common approach to this case is to fit a mixture of 2 normal or lognormal distributions. Further discussion of fitting mixtures of distributions is beyond the scope of this Handbook.

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.3. Graphical Techniques: Alphabetic
1.3.3.14. Histogram

1.3.3.14.5. Histogram Interpretation: Bimodal Mixture of 2 Normals

Histogram from Mixture of 2 Normal Distributions



Discussion of Unimodal and Bimodal

The histogram shown above illustrates data from a bimodal (2 peak) distribution.

In contrast to the previous example, this example illustrates bimodality due not to an underlying deterministic model, but bimodality due to a mixture of probability models. In this case, each of the modes appears to have a rough bell-shaped component. One could easily imagine the above histogram being generated by a process consisting of two normal distributions with the same standard deviation but with two different locations (one centered at approximately 9.17 and the other centered at approximately 9.26). If this is the case, then the research challenge is to determine physically why there are two similar but separate sub-processes.

Recommended Next Steps

If the histogram indicates that the data might be appropriately fit with a mixture of two normal distributions, the recommended next step is:

Fit the normal mixture model using either least squares or maximum likelihood. The general normal mixing model is

$$M = p\phi_1 + (1 - p)\phi_2$$

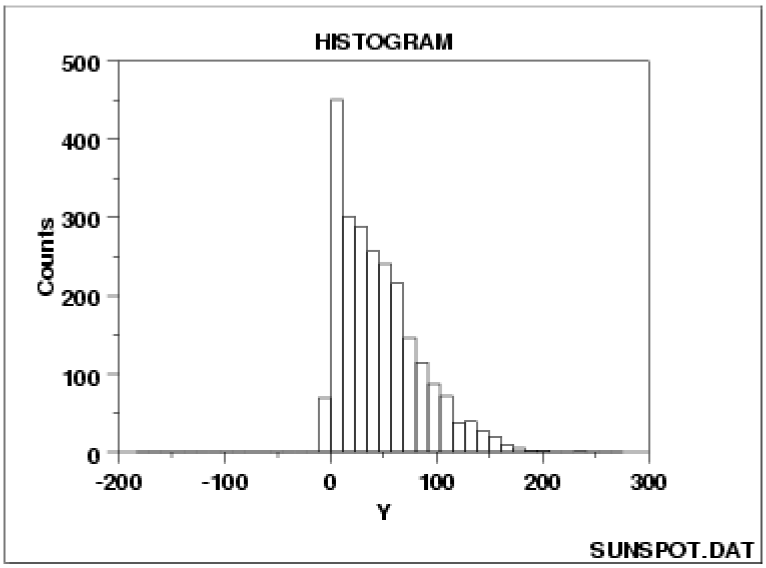
where p is the mixing proportion (between 0 and 1) and ϕ_1 and ϕ_2 are normal probability density functions with location and scale parameters μ_1, σ_1, μ_2, and σ_2, respectively. That is, there are 5 parameters to estimate in the fit.

Whether maximum likelihood or least squares is used, the quality of the fit is sensitive to good starting values. For the mixture of two normals, the histogram can be used to provide initial estimates for the location and scale parameters of the two normal distributions.

Both Dataplot code and R code can be used to fit a mixture of two normals.

1.3.3.14.6. Histogram Interpretation: Skewed (Non-Normal) Right

Right-Skewed Histogram



Discussion of Skewness

A symmetric distribution is one in which the 2 "halves" of the histogram appear as mirror-images of one another. A skewed (non-symmetric) distribution is a distribution in which there is no such mirror-imaging.

For skewed distributions, it is quite common to have one tail of the distribution considerably longer or drawn out relative to the other tail. A "skewed right" distribution is one in which the tail is on the right side. A "skewed left" distribution is one in which the tail is on the left side. The above histogram is for a distribution that is skewed right.

Skewed distributions bring a certain philosophical complexity to the very process of estimating a "typical value" for the distribution. To be specific, suppose that the analyst has a collection of 100 values randomly drawn from a distribution, and wishes to summarize these 100 observations by a "typical value". What does typical value mean? If the distribution is symmetric, the typical value is unambiguous-- it is a well-defined center of the distribution. For example, for a bell-shaped symmetric distribution, a center point is identical to that value at the peak of the distribution.

For a skewed distribution, however, there is no "center" in the usual sense of the word. Be that as it may, several "typical value" metrics are often used for skewed distributions. The first metric is the mode of the distribution. Unfortunately, for severely-skewed distributions, the mode may be at or near the left or right tail of the data and so it seems not to be a good representative of the center of the distribution. As a second choice, one could conceptually argue that the mean (the point on the horizontal axis where the distributiuon would balance) would serve well as the typical value. As a third choice, others may argue that the median (that value on the horizontal axis which has exactly 50% of the data to the left (and also to the right) would serve as a good typical value.

For symmetric distributions, the conceptual problem disappears because at the population level the mode, mean, and median are identical. For skewed distributions, however, these 3 metrics are markedly different. In practice, for skewed distributions the most commonly reported typical value is the mean; the next most common is the median; the least common is the mode. Because each of these 3 metrics reflects a different aspect of "centerness", it is recommended that the analyst report at least 2 (mean and median), and preferably all 3 (mean, median, and mode) in summarizing and characterizing a data set.

Some Causes for Skewed Data

Skewed data often occur due to lower or upper bounds on the data. That is, data that have a lower bound are often skewed right while data that have an upper bound are often skewed left. Skewness can also result from start-up effects. For example, in reliability applications some processes may have a large number of initial failures that could cause left skewness. On the other hand, a reliability process could have a long start-up period where failures are rare resulting in right-skewed data.

Data collected in scientific and engineering applications often have a lower bound of zero. For example, failure data must be non-negative. Many measurement processes generate only positive data. Time to occurence and size are common measurements that cannot be less than zero.

Recommended Next Steps

If the histogram indicates a right-skewed data set, the recommended next steps are to:

1. Quantitatively summarize the data by computing and reporting the sample mean, the sample median, and the sample mode.

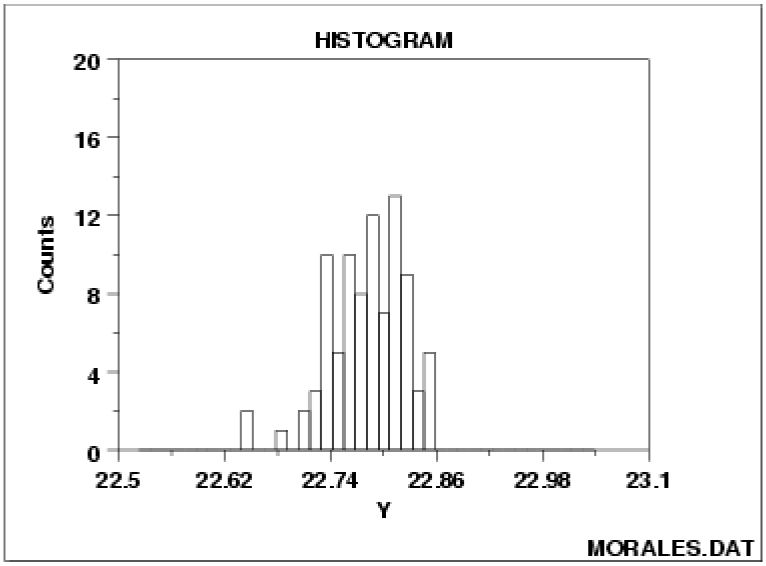
2. Determine the best-fit distribution (skewed-right)

from the
- Weibull family (for the maximum)
- Gamma family
- Chi-square family
- Lognormal family
- Power lognormal family

3. Consider a normalizing transformation such as the Box-Cox transformation.

1.3.3.14.7. Histogram Interpretation: Skewed (Non-Symmetric) Left

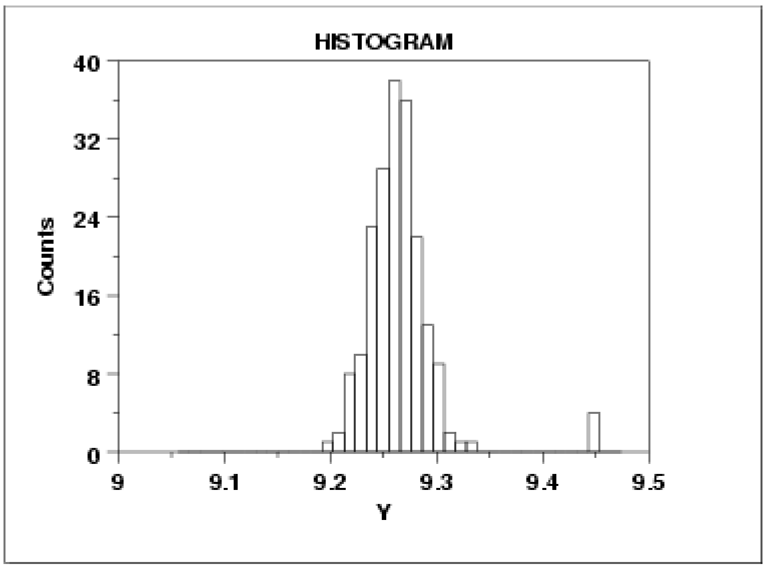
Skewed Left Histogram



The issues for skewed left data are similar to those for skewed right data.

1.3.3.14.8. Histogram Interpretation: Symmetric with Outlier

Symmetric Histogram with Outlier



Discussion of Outliers

A symmetric distribution is one in which the 2 "halves" of the histogram appear as mirror-images of one another. The above example is symmetric with the exception of outlying data near Y = 4.5.

An outlier is a data point that comes from a distribution different (in location, scale, or distributional form) from the bulk of the data. In the real world, outliers have a range of causes, from as simple as

1. operator blunders
2. equipment failures
3. day-to-day effects
4. batch-to-batch differences
5. anomalous input conditions
6. warm-up effects

to more subtle causes such as

1. A change in settings of factors that (knowingly or unknowingly) affect the response.
2. Nature is trying to tell us something.

Outliers Should be Investigated

All outliers should be taken seriously and should be investigated thoroughly for explanations. Automatic outlier-rejection schemes (such as throw out all data beyond 4 sample standard deviations from the sample mean) are particularly dangerous.

The classic case of automatic outlier rejection becoming automatic information rejection was the South Pole ozone depletion problem. Ozone depletion over the South Pole would have been detected years earlier except for the fact that the satellite data recording the low ozone readings had outlier-rejection code that automatically screened out the "outliers" (that is, the low ozone readings) before the analysis was conducted. Such inadvertent (and incorrect) purging went on for years. It was not until ground-based South Pole readings started detecting low ozone readings that someone decided to double-check as to why the satellite had not picked up this fact--it had, but it had gotten thrown out!

The best attitude is that outliers are our "friends", outliers are trying to tell us something, and we should not stop until we are comfortable in the explanation for each outlier.

Recommended Next Steps

If the histogram shows the presence of outliers, the recommended next steps are:

1. Graphically check for outliers (in the commonly encountered normal case) by generating a box plot. In general, box plots are a much better graphical tool for detecting outliers than are histograms.

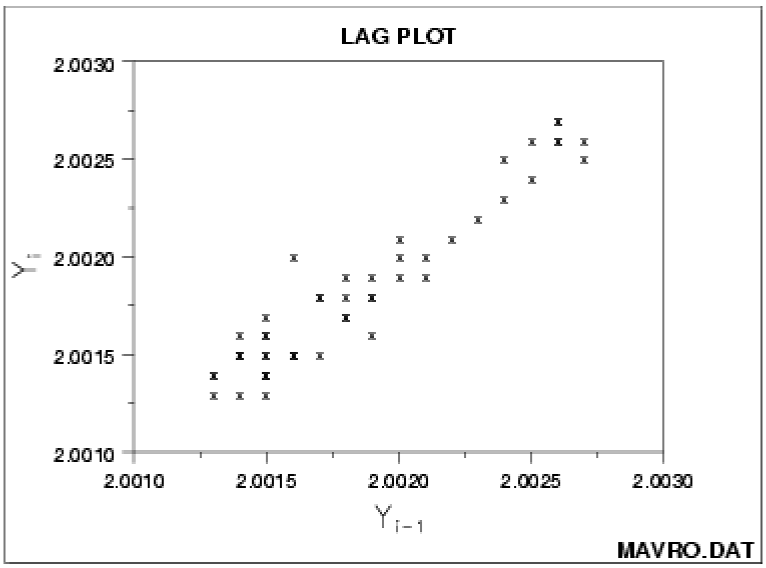
2. Quantitatively check for outliers (in the commonly encountered normal case) by carrying out Grubbs test which indicates how many sample standard deviations away from the sample mean are the data in question. Large values indicate outliers.

1.3.3.15. Lag Plot

Purpose: Check for randomness

A lag plot checks whether a data set or time series is random or not. Random data should not exhibit any identifiable structure in the lag plot. Non-random structure in the lag plot indicates that the underlying data are not random. Several common patterns for lag plots are shown in the examples below.

Sample Plot



This sample lag plot exhibits a linear pattern. This shows that the data are strongly non-random and further suggests that an autoregressive model might be appropriate.

Definition

A lag is a fixed time displacement. For example, given a data set Y_1, Y_2 ..., Y_n, Y_2 and Y_7 have lag 5 since 7 - 2 = 5. Lag plots can be generated for any arbitrary lag, although the most commonly used lag is 1.

A plot of lag 1 is a plot of the values of Y_i versus Y_{i-1}

- Vertical axis: Y_i for all i
- Horizontal axis: Y_{i-1} for all i

Questions

Lag plots can provide answers to the following questions:

1. Are the data random?
2. Is there serial correlation in the data?
3. What is a suitable model for the data?
4. Are there outliers in the data?

Importance

Inasmuch as randomness is an underlying assumption for most statistical estimation and testing techniques, the lag plot should be a routine tool for researchers.

Examples

- Random (White Noise)
- Weak autocorrelation
- Strong autocorrelation and autoregressive model
- Sinusoidal model and outliers

Related Techniques

Autocorrelation Plot
Spectrum
Runs Test

Case Study

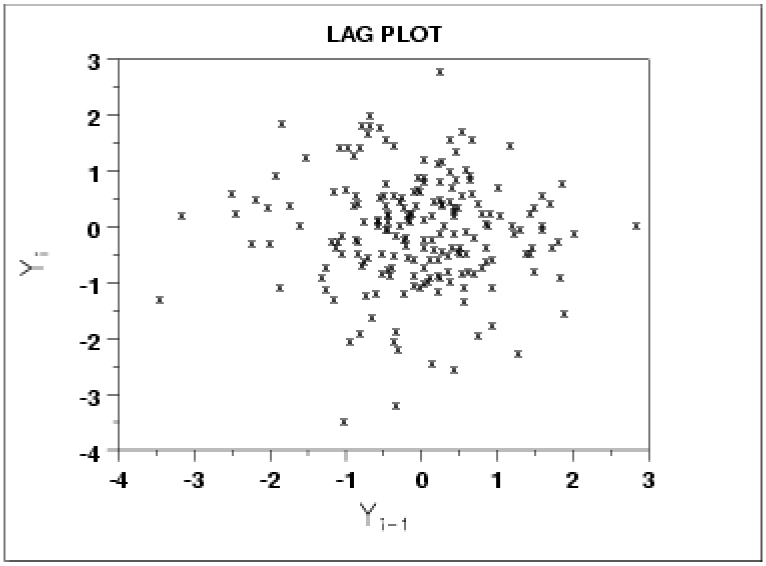
The lag plot is demonstrated in the beam deflection data case study.

Software

Lag plots are not directly available in most general purpose statistical software programs. Since the lag plot is essentially a scatter plot with the 2 variables properly lagged, it should be feasible to write a macro for the lag plot in most statistical programs.

1.3.3.15.1. Lag Plot: Random Data

Lag Plot



Conclusions

We can make the following conclusions based on the above plot.

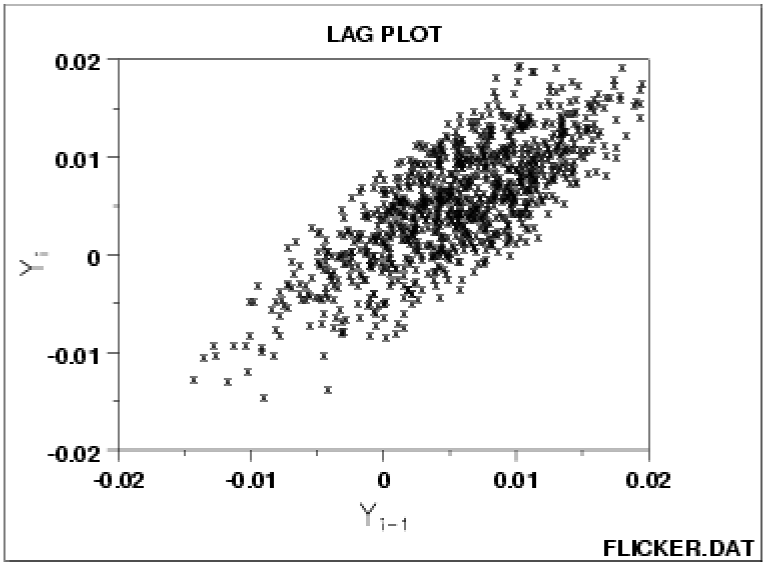
1. The data are random.
2. The data exhibit no autocorrelation.
3. The data contain no outliers.

Discussion

The lag plot shown above is for lag = 1. Note the absence of structure. One cannot infer, from a current value Y_{i-1}, the next value Y_i. Thus for a known value Y_{i-1} on the horizontal axis (say, Y_{i-1} = +0.5), the Y_i-th value could be virtually anything (from Y_i = -2.5 to Y_i = +1.5). Such non-association is the essence of randomness.

1.3.3.15.2. Lag Plot: Moderate Autocorrelation

Lag Plot



Conclusions

We can make the conclusions based on the above plot.

1. The data are from an underlying autoregressive model with moderate positive autocorrelation
2. The data contain no outliers.

Discussion

In the plot above for lag = 1, note how the points tend to cluster (albeit noisily) along the diagonal. Such clustering is the lag plot signature of moderate autocorrelation.

If the process were completely random, knowledge of a current observation (say Y_{i-1} = 0) would yield virtually no knowledge about the next observation Y_i. If the process has moderate autocorrelation, as above, and if Y_{i-1} = 0, then the range of possible values for Y_i is seen to be restricted to a smaller range (.01 to +.01). This suggests prediction is possible using an autoregressive model.

Recommended Next Step

Estimate the parameters for the autoregressive model:

$$Y_i = A_0 + A_1 * Y_{i-1} + E_i$$

Since ***Y*** and ***Y*** are precisely the axes of the lag plot,

i ***i-1***

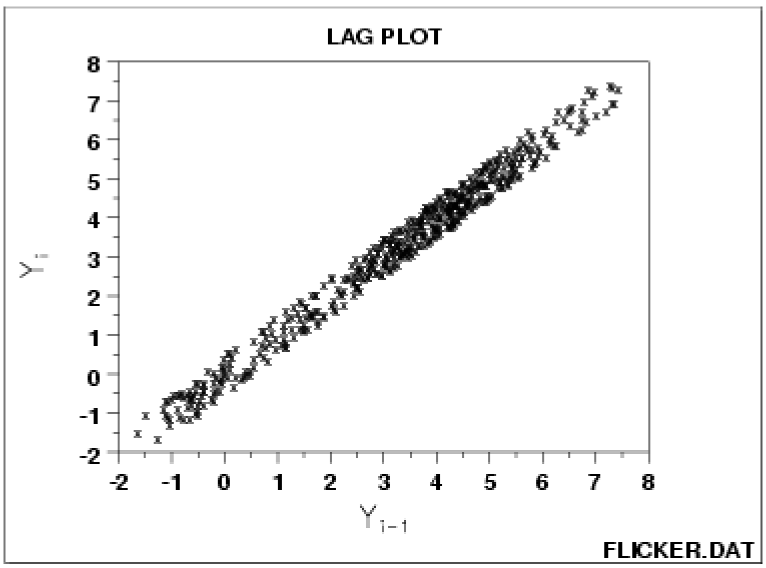
such estimation is a linear regression straight from the lag plot.

The residual standard deviation for the autoregressive model will be much smaller than the residual standard deviation for the default model

$$Y_i = A_0 + E_i$$

1.3.3.15.3. Lag Plot: Strong Autocorrelation and Autoregressive Model

Lag Plot



Conclusions

We can make the following conclusions based on the above plot.

1. The data come from an underlying autoregressive model with strong positive autocorrelation
2. The data contain no outliers.

Discussion

Note the tight clustering of points along the diagonal. This is the lag plot signature of a process with strong positive autocorrelation. Such processes are highly non-random--there is strong association between an observation and a succeeding observation. In short, if you know Y_{i-1} you can make a strong guess as to what Y_i will be.

If the above process were completely random, the plot would have a shotgun pattern, and knowledge of a current observation (say Y_{i-1} = 3) would yield virtually no knowledge about the next observation Y_i (it could here be anywhere from -2 to +8). On the other hand, if the process had strong autocorrelation, as seen above, and if Y_{i-1} = 3, then the range of possible values for Y is seen to be

i
restricted to a smaller range (2 to 4)--still wide, but an improvement nonetheless (relative to -2 to +8) in predictive power.

Recommended Next Step

When the lag plot shows a strongly autoregressive pattern and only successive observations appear to be correlated, the next steps are to:

1. Extimate the parameters for the autoregressive model:

   $$Y_i = A_0 + A_1 * Y_{i-1} + E_i$$

   Since Y_i and Y_{i-1} are precisely the axes of the lag plot, such estimation is a linear regression straight from the lag plot.

   The residual standard deviation for this autoregressive model will be much smaller than the residual standard deviation for the default model

   $$Y_i = A_0 + E_i$$

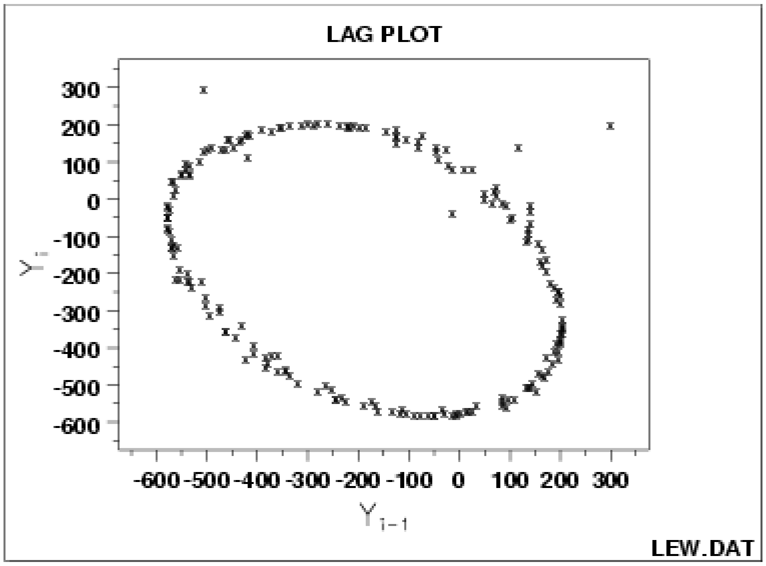
2. Reexamine the system to arrive at an explanation for the strong autocorrelation. Is it due to the

   1. phenomenon under study; or
   2. drifting in the environment; or
   3. contamination from the data acquisition system?

   Sometimes the source of the problem is contamination and carry-over from the data acquisition system where the system does not have time to electronically recover before collecting the next data point. If this is the case, then consider slowing down the sampling rate to achieve randomness.

1.3.3.15.4. Lag Plot: Sinusoidal Models and Outliers

Lag Plot



Conclusions

We can make the following conclusions based on the above plot.

1. The data come from an underlying single-cycle sinusoidal model.
2. The data contain three outliers.

Discussion

In the plot above for lag = 1, note the tight elliptical clustering of points. Processes with a single-cycle sinusoidal model will have such elliptical lag plots.

Consequences of Ignoring Cyclical Pattern

If one were to naively assume that the above process came from the null model

$$Y_i = A_0 + E_i$$

and then estimate the constant by the sample mean, then the analysis would suffer because

1. the sample mean would be biased and meaningless;
2. the confidence limits would be meaningless and optimistically small.

The proper model

$$Y_i = C + \alpha \sin(2\pi\omega t_i + \phi) + E_i$$

(where α is the amplitude, ω is the frequency--between 0 and .5 cycles per observation--, and ϕ is the phase) can be fit by standard non-linear least squares, to estimate the coefficients and their uncertainties.

The lag plot is also of value in outlier detection. Note in the above plot that there appears to be 4 points lying off the ellipse. However, in a lag plot, each point in the original data set Y shows up twice in the lag plot--once as Y_i and once as Y_{i-1}. Hence the outlier in the upper left at Y_i = 300 is the same raw data value that appears on the far right at Y_{i-1} = 300. Thus (-500,300) and (300,200) are due to the same outlier, namely the 158th data point: 300. The correct value for this 158th point should be approximately -300 and so it appears that a sign got dropped in the data collection. The other two points lying off the ellipse, at roughly (100,100) and at (0,-50), are caused by two faulty data values: the third data point of -15 should be about +125 and the fourth data point of +141 should be about -50, respectively. Hence the 4 apparent lag plot outliers are traceable to 3 actual outliers in the original run sequence: at points 4 (-15), 5 (141) and 158 (300). In retrospect, only one of these (point 158 (= 300)) is an obvious outlier in the run sequence plot.

Unexpected Value of EDA

Frequently a technique (e.g., the lag plot) is constructed to check one aspect (e.g., randomness) which it does well. Along the way, the technique also highlights some other anomaly of the data (namely, that there are 3 outliers). Such outlier identification and removal is extremely important for detecting irregularities in the data collection system, and also for arriving at a "purified" data set for modeling. The lag plot plays an important role in such outlier identification.

Recommended Next Step

When the lag plot indicates a sinusoidal model with possible outliers, the recommended next steps are:

1. Do a spectral plot to obtain an initial estimate of the frequency of the underlying cycle. This will be helpful as a starting value for the subsequent non-linear fitting.

2. Omit the outliers.

3. Carry out a non-linear fit of the model to the 197 points.

$$Y_i = C + \alpha \sin(2\pi\omega t_i + \phi) + E_i$$

1.3.3.16. Linear Correlation Plot

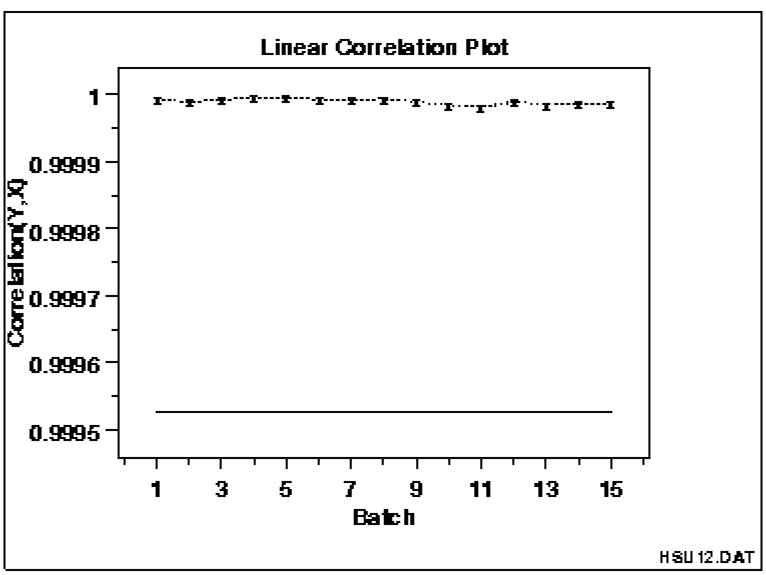
Purpose: Detect changes in correlation between groups

Linear correlation plots are used to assess whether or not correlations are consistent across groups. That is, if your data is in groups, you may want to know if a single correlation can be used across all the groups or whether separate correlations are required for each group.

Linear correlation plots are often used in conjunction with linear slope, linear intercept, and linear residual standard deviation plots. A linear correlation plot could be generated intially to see if linear fitting would be a fruitful direction. If the correlations are high, this implies it is worthwhile to continue with the linear slope, intercept, and residual standard deviation plots. If the correlations are weak, a different model needs to be pursued.

In some cases, you might not have groups. Instead you may have different data sets and you want to know if the same correlation can be adequately applied to each of the data sets. In this case, simply think of each distinct data set as a group and apply the linear slope plot as for groups.

Sample Plot



This linear correlation plot shows that the correlations are high for all groups. This implies that linear fits could provide a good model for each of these groups.

Definition: Group Correlations Versus Group ID

Linear correlation plots are formed by:

- Vertical axis: Group correlations
- Horizontal axis: Group identifier

A reference line is plotted at the correlation between the full data sets.

Questions

The linear correlation plot can be used to answer the following questions.

1. Are there linear relationships across groups?
2. Are the strength of the linear relationships relatively constant across the groups?

Importance: Checking Group Homogeneity

For grouped data, it may be important to know whether the different groups are homogeneous (i.e., similar) or heterogeneous (i.e., different). Linear correlation plots help answer this question in the context of linear fitting.

Related Techniques

Linear Intercept Plot
Linear Slope Plot
Linear Residual Standard Deviation Plot
Linear Fitting

Case Study

The linear correlation plot is demonstrated in the Alaska pipeline data case study.

Software

Most general purpose statistical software programs do not support a linear correlation plot. However, if the statistical program can generate correlations over a group, it should be feasible to write a macro to generate this plot.

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.3.3.17. Linear Intercept Plot

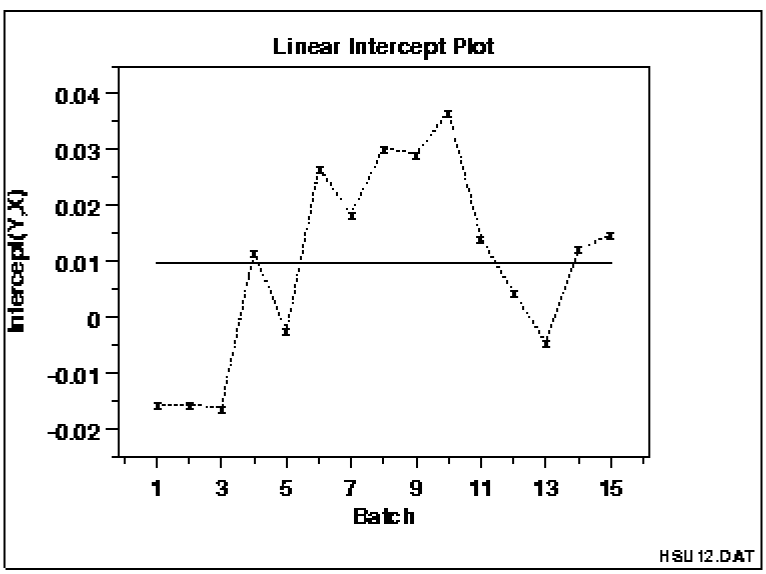
Purpose: Detect changes in linear intercepts between groups

Linear intercept plots are used to graphically assess whether or not linear fits are consistent across groups. That is, if your data have groups, you may want to know if a single fit can be used across all the groups or whether separate fits are required for each group.

Linear intercept plots are typically used in conjunction with linear slope and linear residual standard deviation plots.

In some cases you might not have groups. Instead, you have different data sets and you want to know if the same fit can be adequately applied to each of the data sets. In this case, simply think of each distinct data set as a group and apply the linear intercept plot as for groups.

Sample Plot



This linear intercept plot shows that there is a shift in intercepts. Specifically, the first three intercepts are lower than the intercepts for the other groups. Note that these are small differences in the intercepts.

Definition: Group Intercepts Versus

Linear intercept plots are formed by:

- Vertical axis: Group intercepts from linear fits
- Horizontal axis: Group identifier

Group ID

A reference line is plotted at the intercept from a linear fit using all the data.

Questions

The linear intercept plot can be used to answer the following questions.

1. Is the intercept from linear fits relatively constant across groups?
2. If the intercepts vary across groups, is there a discernible pattern?

Importance: Checking Group Homogeneity

For grouped data, it may be important to know whether the different groups are homogeneous (i.e., similar) or heterogeneous (i.e., different). Linear intercept plots help answer this question in the context of linear fitting.

Related Techniques

Linear Correlation Plot
Linear Slope Plot
Linear Residual Standard Deviation Plot
Linear Fitting

Case Study

The linear intercept plot is demonstrated in the Alaska pipeline data case study.

Software

Most general purpose statistical software programs do not support a linear intercept plot. However, if the statistical program can generate linear fits over a group, it should be feasible to write a macro to generate this plot.

1.3.3.18. Linear Slope Plot

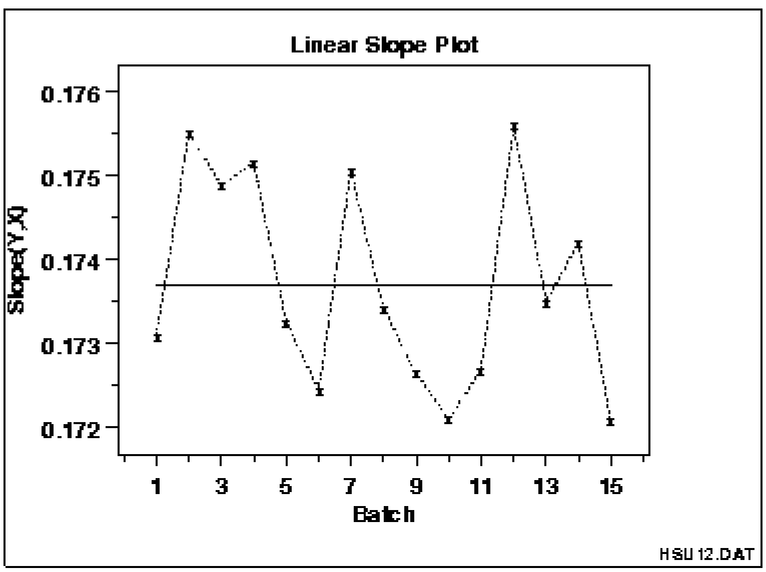
Purpose: Detect changes in linear slopes between groups

Linear slope plots are used to graphically assess whether or not linear fits are consistent across groups. That is, if your data have groups, you may want to know if a single fit can be used across all the groups or whether separate fits are required for each group.

Linear slope plots are typically used in conjunction with linear intercept and linear residual standard deviation plots.

In some cases you might not have groups. Instead, you have different data sets and you want to know if the same fit can be adequately applied to each of the data sets. In this case, simply think of each distinct data set as a group and apply the linear slope plot as for groups.

Sample Plot



This linear slope plot shows that the slopes are about 0.174 (plus or minus 0.002) for all groups. There does not appear to be a pattern in the variation of the slopes. This implies that a single fit may be adequate.

Definition: Group Slopes Versus Group ID

Linear slope plots are formed by:

- Vertical axis: Group slopes from linear fits
- Horizontal axis: Group identifier

A reference line is plotted at the slope from a linear fit using all the data.

Questions

The linear slope plot can be used to answer the following questions.

1. Do you get the same slope across groups for linear fits?
2. If the slopes differ, is there a discernible pattern in the slopes?

Importance: Checking Group Homogeneity

For grouped data, it may be important to know whether the different groups are homogeneous (i.e., similar) or heterogeneous (i.e., different). Linear slope plots help answer this question in the context of linear fitting.

Related Techniques

Linear Intercept Plot
Linear Correlation Plot
Linear Residual Standard Deviation Plot
Linear Fitting

Case Study

The linear slope plot is demonstrated in the Alaska pipeline data case study.

Software

Most general purpose statistical software programs do not support a linear slope plot. However, if the statistical program can generate linear fits over a group, it should be feasible to write a macro to generate this plot.

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.3.3.19. Linear Residual Standard Deviation Plot

Purpose: Detect Changes in Linear Residual Standard Deviation Between Groups

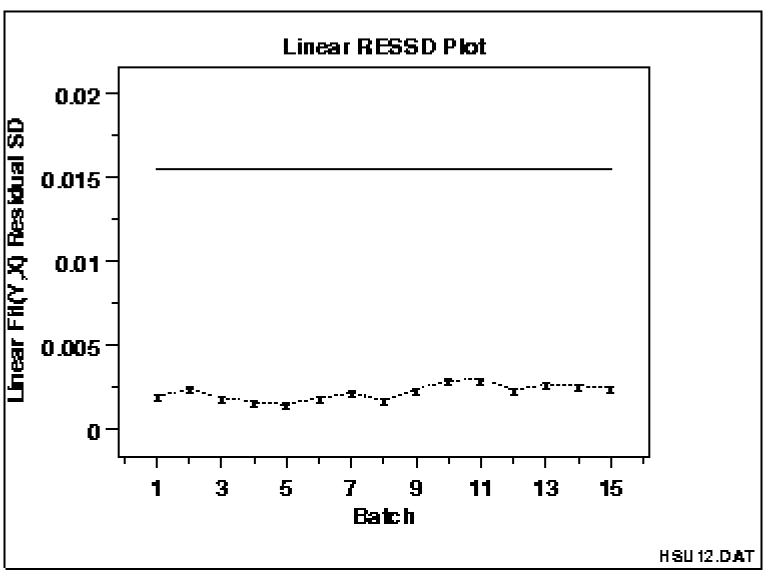
Linear residual standard deviation (RESSD) plots are used to graphically assess whether or not linear fits are consistent across groups. That is, if your data have groups, you may want to know if a single fit can be used across all the groups or whether separate fits are required for each group.

The residual standard deviation is a goodness-of-fit measure. That is, the smaller the residual standard deviation, the closer is the fit to the data.

Linear RESSD plots are typically used in conjunction with linear intercept and linear slope plots. The linear intercept and slope plots convey whether or not the fits are consistent across groups while the linear RESSD plot conveys whether the adequacy of the fit is consistent across groups.

In some cases you might not have groups. Instead, you have different data sets and you want to know if the same fit can be adequately applied to each of the data sets. In this case, simply think of each distinct data set as a group and apply the linear RESSD plot as for groups.

Sample Plot



This linear RESSD plot shows that the residual standard deviations from a linear fit are about 0.0025 for all the groups.

Definition: Group Residual Standard Deviation Versus Group ID

Linear RESSD plots are formed by:

- Vertical axis: Group residual standard deviations from linear fits
- Horizontal axis: Group identifier

A reference line is plotted at the residual standard deviation from a linear fit using all the data. This reference line will typically be much greater than any of the individual residual standard deviations.

Questions

The linear RESSD plot can be used to answer the following questions.

1. Is the residual standard deviation from a linear fit constant across groups?
2. If the residual standard deviations vary, is there a discernible pattern across the groups?

Importance: Checking Group Homogeneity

For grouped data, it may be important to know whether the different groups are homogeneous (i.e., similar) or heterogeneous (i.e., different). Linear RESSD plots help answer this question in the context of linear fitting.

Related Techniques

Linear Intercept Plot
Linear Slope Plot
Linear Correlation Plot
Linear Fitting

Case Study

The linear residual standard deviation plot is demonstrated in the Alaska pipeline data case study.

Software

Most general purpose statistical software programs do not support a linear residual standard deviation plot. However, if the statistical program can generate linear fits over a group, it should be feasible to write a macro to generate this plot.

1.3.3.20. Mean Plot

Purpose: Detect changes in location between groups

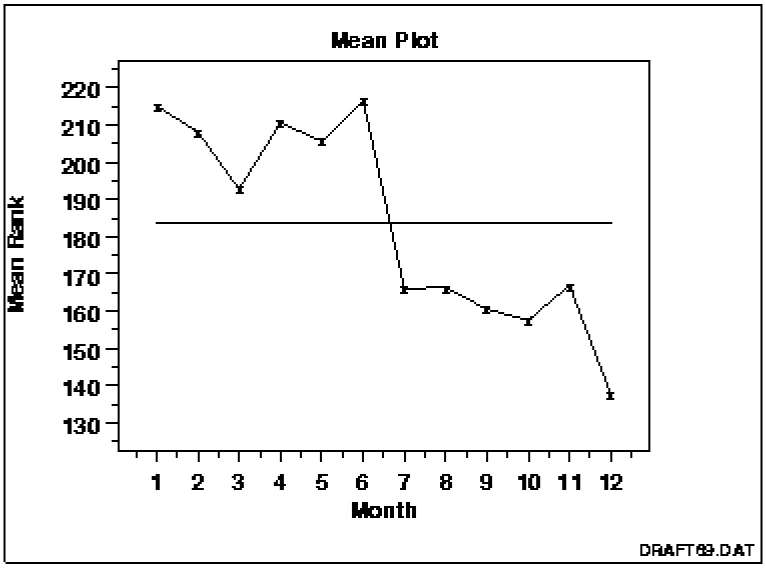
Mean plots are used to see if the mean varies between different groups of the data. The grouping is determined by the analyst. In most cases, the data set contains a specific grouping variable. For example, the groups may be the levels of a factor variable. In the sample plot below, the months of the year provide the grouping.

Mean plots can be used with ungrouped data to determine if the mean is changing over time. In this case, the data are split into an arbitrary number of equal-sized groups. For example, a data series with 400 points can be divided into 10 groups of 40 points each. A mean plot can then be generated with these groups to see if the mean is increasing or decreasing over time.

Although the mean is the most commonly used measure of location, the same concept applies to other measures of location. For example, instead of plotting the mean of each group, the median or the trimmed mean might be plotted instead. This might be done if there were significant outliers in the data and a more robust measure of location than the mean was desired.

Mean plots are typically used in conjunction with standard deviation plots. The mean plot checks for shifts in location while the standard deviation plot checks for shifts in scale.

Sample Plot



This sample mean plot shows a shift of location after the 6th month.

Definition: Group Means Versus Group ID

Mean plots are formed by:

- Vertical axis: Group mean
- Horizontal axis: Group identifier

A reference line is plotted at the overall mean.

Questions

The mean plot can be used to answer the following questions.

1. Are there any shifts in location?
2. What is the magnitude of the shifts in location?
3. Is there a distinct pattern in the shifts in location?

Importance: Checking Assumptions

A common assumption in 1-factor analyses is that of constant location. That is, the location is the same for different levels of the factor variable. The mean plot provides a graphical check for that assumption. A common assumption for univariate data is that the location is constant. By grouping the data into equal intervals, the mean plot can provide a graphical test of this assumption.

Related Techniques

Standard Deviation Plot
DOE Mean Plot
Box Plot

Software

Most general purpose statistical software programs do not support a mean plot. However, if the statistical program can generate the mean over a group, it should be feasible to write a macro to generate this plot.

1.3.3.21. Normal Probability Plot

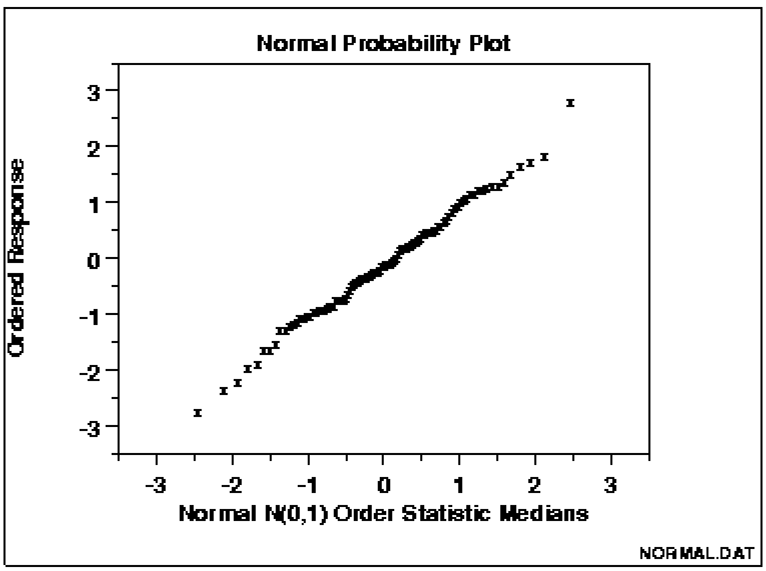
Purpose: Check If Data Are Approximately Normally Distributed

The normal probability plot (Chambers 1983) is a graphical technique for assessing whether or not a data set is approximately normally distributed.

The data are plotted against a theoretical normal distribution in such a way that the points should form an approximate straight line. Departures from this straight line indicate departures from normality.

The normal probability plot is a special case of the probability plot. We cover the normal probability plot separately due to its importance in many applications.

Sample Plot



The points on this plot form a nearly linear pattern, which indicates that the normal distribution is a good model for this data set.

Definition: Ordered Response Values Versus Normal Order Statistic Medians

The normal probability plot is formed by:

- Vertical axis: Ordered response values
- Horizontal axis: Normal order statistic medians

The observations are plotted as a function of the corresponding normal order statistic medians which are defined as:

$$N(i) = G(U(i))$$

where U(i) are the uniform order statistic medians (defined below) and G is the percent point function of the normal distribution. The percent point function is the inverse of the cumulative distribution function (probability that x is less than or equal to some value). That is, given a probability, we want the corresponding x of the cumulative distribution function.

The uniform order statistic medians are defined as:

$$U(i) = 1 - U(n) \text{ for } i = 1$$
$$U(i) = (i - 0.3175)/(n + 0.365) \text{ for } i = 2, 3, ..., n-1$$
$$U(i) = 0.5^{(1/n)} \text{ for } i = n$$

In addition, a straight line can be fit to the points and added as a reference line. The further the points vary from this line, the greater the indication of departures from normality.

Probability plots for distributions other than the normal are computed in exactly the same way. The normal percent point function (the G) is simply replaced by the percent point function of the desired distribution. That is, a probability plot can easily be generated for any distribution for which you have the percent point function.

One advantage of this method of computing probability plots is that the intercept and slope estimates of the fitted line are in fact estimates for the location and scale parameters of the distribution. Although this is not too important for the normal distribution since the location and scale are estimated by the mean and standard deviation, respectively, it can be useful for many other distributions.

The correlation coefficient of the points on the normal probability plot can be compared to a table of critical values to provide a formal test of the hypothesis that the data come from a normal distribution.

Questions

The normal probability plot is used to answer the following questions.

1. Are the data normally distributed?
2. What is the nature of the departure from normality (data skewed, shorter than expected tails, longer than expected tails)?

Importance: Check

The underlying assumptions for a measurement process are that the data should behave like:

Normality Assumption

1. random drawings;
2. from a fixed distribution;
3. with fixed location;
4. with fixed scale.

Probability plots are used to assess the assumption of a fixed distribution. In particular, most statistical models are of the form:

response = deterministic + random

where the deterministic part is the fit and the random part is error. This error component in most common statistical models is specifically assumed to be normally distributed with fixed location and scale. This is the most frequent application of normal probability plots. That is, a model is fit and a normal probability plot is generated for the residuals from the fitted model. If the residuals from the fitted model are not normally distributed, then one of the major assumptions of the model has been violated.

Examples

1. Data are normally distributed
2. Data have short tails
3. Data have fat tails
4. Data are skewed right

Related Techniques

Histogram
Probability plots for other distributions (e.g., Weibull)
Probability plot correlation coefficient plot (PPCC plot)
Anderson-Darling Goodness-of-Fit Test
Chi-Square Goodness-of-Fit Test
Kolmogorov-Smirnov Goodness-of-Fit Test

Case Study

The normal probability plot is demonstrated in the heat flow meter data case study.

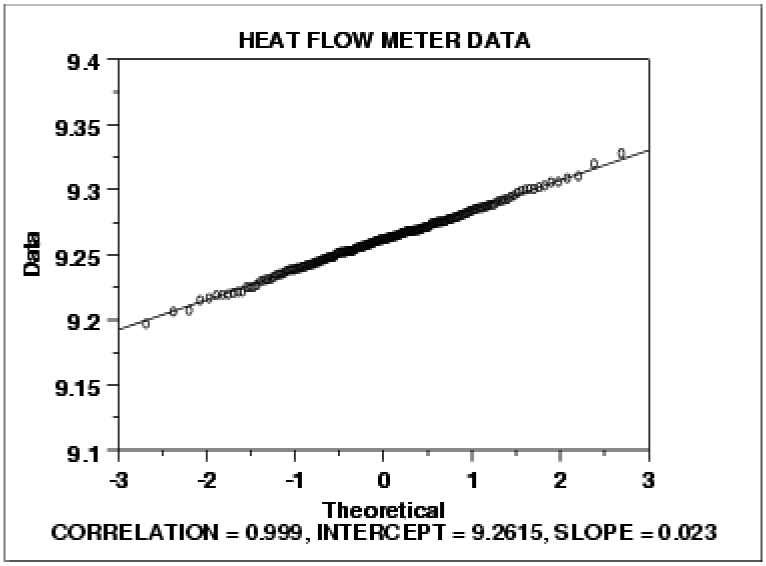
Software

Most general purpose statistical software programs can generate a normal probability plot.

1.3.3.21.1. Normal Probability Plot: Normally Distributed Data

Normal Probability Plot

The following normal probability plot is from the heat flow meter data.



Conclusions

We can make the following conclusions from the above plot.

1. The normal probability plot shows a strongly linear pattern. There are only minor deviations from the line fit to the points on the probability plot.
2. The normal distribution appears to be a good model for these data.

Discussion

Visually, the probability plot shows a strongly linear pattern. This is verified by the correlation coefficient of 0.9989 of the line fit to the probability plot. The fact that the points in the lower and upper extremes of the plot do not deviate significantly from the straight-line pattern indicates that there are not any significant outliers (relative to a normal distribution).

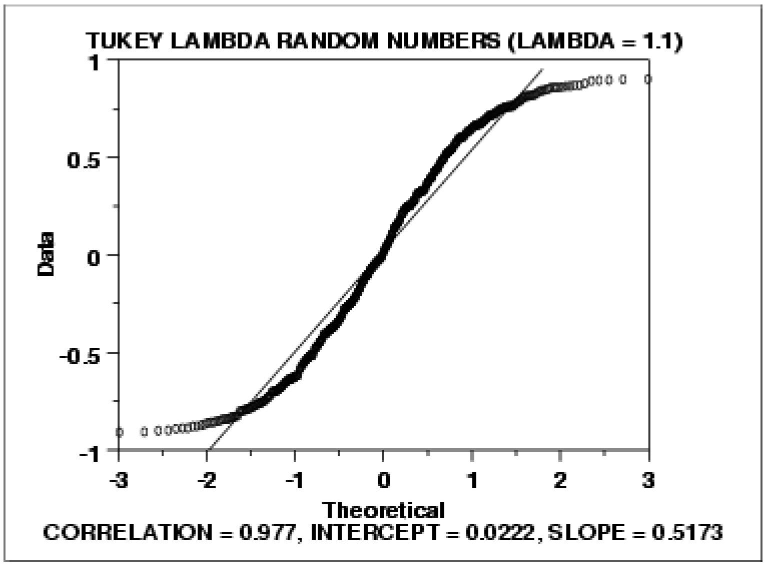
In this case, we can quite reasonably conclude that the normal distribution provides an excellent model for the data. The intercept and slope of the fitted line give estimates of

9.26 and 0.023 for the location and scale parameters of the fitted normal distribution.

1.3.3.21.2. Normal Probability Plot: Data Have Short Tails

Normal Probability Plot for Data with Short Tails

The following is a normal probability plot for 500 random numbers generated from a Tukey-Lambda distribution with the λ parameter equal to 1.1.



Conclusions

We can make the following conclusions from the above plot.

1. The normal probability plot shows a non-linear pattern.
2. The normal distribution is not a good model for these data.

Discussion

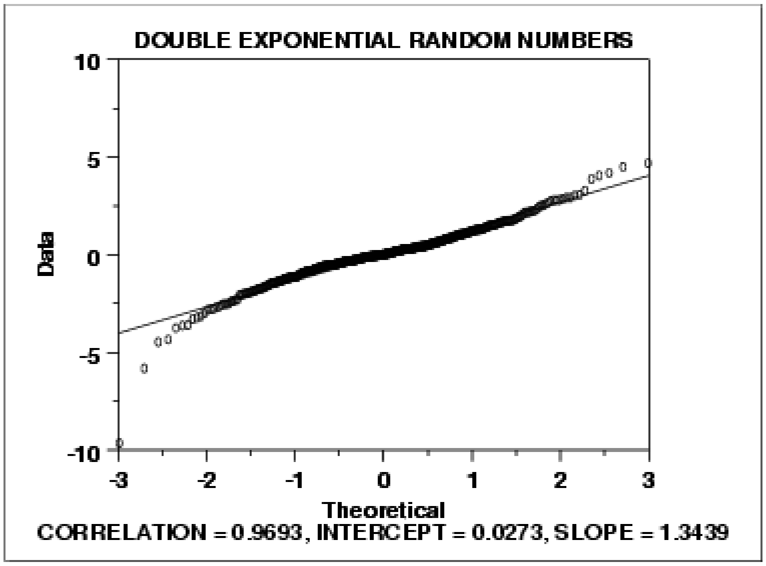
For data with short tails relative to the normal distribution, the non-linearity of the normal probability plot shows up in two ways. First, the middle of the data shows an S-like pattern. This is common for both short and long tails. Second, the first few and the last few points show a marked departure from the reference fitted line. In comparing this plot to the long tail example in the next section, the important difference is the direction of the departure from the fitted line for the first few and last few points. For short tails, the first few points show increasing departure from the fitted line *above* the line and last few points show increasing departure from the fitted line *below* the line. For long tails, this pattern is reversed.

In this case, we can reasonably conclude that the normal distribution does not provide an adequate fit for this data set. For probability plots that indicate short-tailed distributions, the next step might be to generate a Tukey Lambda PPCC plot. The Tukey Lambda PPCC plot can often be helpful in identifying an appropriate distributional family.

1.3.3.21.3. Normal Probability Plot: Data Have Long Tails

Normal Probability Plot for Data with Long Tails

The following is a normal probability plot of 500 numbers generated from a double exponential distribution. The double exponential distribution is symmetric, but relative to the normal it declines rapidly and has longer tails.



Conclusions

We can make the following conclusions from the above plot.

1. The normal probability plot shows a reasonably linear pattern in the center of the data. However, the tails, particularly the lower tail, show departures from the fitted line.
2. A distribution other than the normal distribution would be a good model for these data.

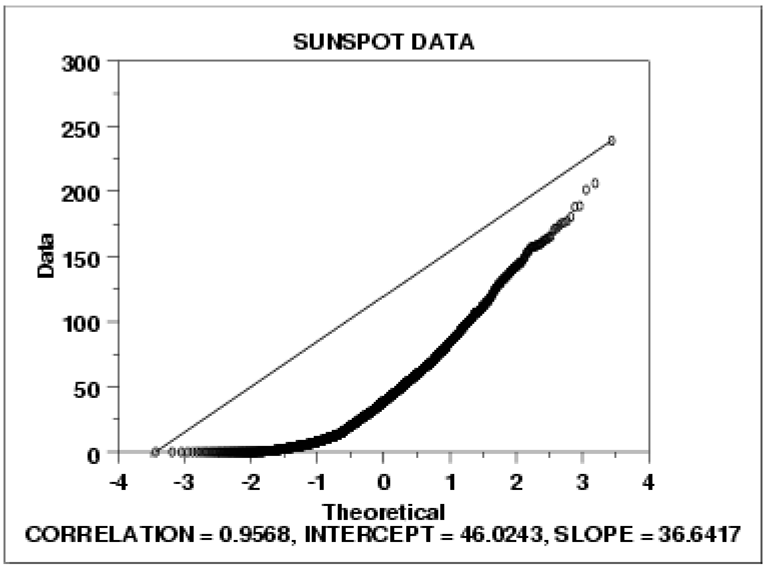
Discussion

For data with long tails relative to the normal distribution, the non-linearity of the normal probability plot can show up in two ways. First, the middle of the data may show an S-like pattern. This is common for both short and long tails. In this particular case, the S pattern in the middle is fairly mild. Second, the first few and the last few points show marked departure from the reference fitted line. In the plot above, this is most noticeable for the first few data points. In comparing this plot to the short-tail example in the previous section, the important difference is the direction of the departure from the fitted line for the first few and the last few points. For long tails, the first few points show increasing departure from the fitted line *below* the line and last few points show increasing departure from the fitted line *above* the line. For short tails, this pattern is reversed.

In this case we can reasonably conclude that the normal distribution can be improved upon as a model for these data. For probability plots that indicate long-tailed distributions, the next step might be to generate a Tukey Lambda PPCC plot. The Tukey Lambda PPCC plot can often be helpful in identifying an appropriate distributional family.

1.3.3.21.4. Normal Probability Plot: Data are Skewed Right

Normal Probability Plot for Data that are Skewed Right



Conclusions We can make the following conclusions from the above plot.

1. The normal probability plot shows a strongly non-linear pattern. Specifically, it shows a quadratic pattern in which all the points are below a reference line drawn between the first and last points.
2. The normal distribution is not a good model for these data.

Discussion This quadratic pattern in the normal probability plot is the signature of a significantly right-skewed data set. Similarly, if all the points on the normal probability plot fell above the reference line connecting the first and last points, that would be the signature pattern for a significantly left-skewed data set.

In this case we can quite reasonably conclude that we need to model these data with a right skewed distribution such as the Weibull or lognormal.

1.3.3.22. Probability Plot

Purpose: Check If Data Follow a Given Distribution

The probability plot (Chambers 1983) is a graphical technique for assessing whether or not a data set follows a given distribution such as the normal or Weibull.

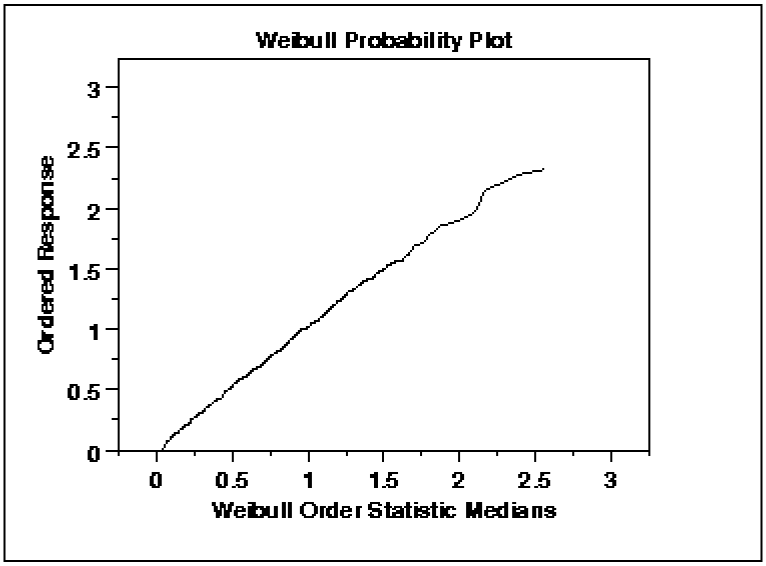
The data are plotted against a theoretical distribution in such a way that the points should form approximately a straight line. Departures from this straight line indicate departures from the specified distribution.

The correlation coefficient associated with the linear fit to the data in the probability plot is a measure of the goodness of the fit. Estimates of the location and scale parameters of the distribution are given by the intercept and slope. Probability plots can be generated for several competing distributions to see which provides the best fit, and the probability plot generating the highest correlation coefficient is the best choice since it generates the straightest probability plot.

For distributions with shape parameters (not counting location and scale parameters), the shape parameters must be known in order to generate the probability plot. For distributions with a single shape parameter, the probability plot correlation coefficient (PPCC) plot provides an excellent method for estimating the shape parameter.

We cover the special case of the normal probability plot separately due to its importance in many statistical applications.

Sample Plot



This data is a set of 500 Weibull random numbers with a shape parameter = 2, location parameter = 0, and scale parameter = 1. The Weibull probability plot indicates that the Weibull distribution does in fact fit these data well.

Definition: Ordered Response Values Versus Order Statistic Medians for the Given Distribution

The probability plot is formed by:

- Vertical axis: Ordered response values
- Horizontal axis: Order statistic medians for the given distribution

The order statistic medians are defined as:

N(i) = G(U(i))

where the U(i) are the uniform order statistic medians (defined below) and G is the percent point function for the desired distribution. The percent point function is the inverse of the cumulative distribution function (probability that *x* is less than or equal to some value). That is, given a probability, we want the corresponding *x* of the cumulative distribution function.

The uniform order statistic medians are defined as:

m(i) = 1 - m(n) for i = 1
m(i) = (i - 0.3175)/(n + 0.365) for i = 2, 3, ..., n-1
m(i) = 0.5**(1/n) for i = n

In addition, a straight line can be fit to the points and added as a reference line. The further the points vary from this line, the greater the indication of a departure from the specified distribution.

This definition implies that a probability plot can be easily generated for any distribution for which the percent point function can be computed.

One advantage of this method of computing proability plots is that the intercept and slope estimates of the fitted line are in fact estimates for the location and scale parameters of the distribution. Although this is not too important for the normal distribution (the location and scale are estimated by the mean and standard deviation, respectively), it can be useful for many other distributions.

Questions

The probability plot is used to answer the following questions:

- Does a given distribution, such as the Weibull, provide a good fit to my data?
- What distribution best fits my data?
- What are good estimates for the location and scale parameters of the chosen distribution?

Importance: Check distributional assumption

The discussion for the normal probability plot covers the use of probability plots for checking the fixed distribution assumption.

Some statistical models assume data have come from a population with a specific type of distribution. For example, in reliability applications, the Weibull, lognormal, and exponential are commonly used distributional models. Probability plots can be useful for checking this distributional assumption.

Related Techniques

Histogram
Probability Plot Correlation Coefficient (PPCC) Plot
Hazard Plot
Quantile-Quantile Plot
Anderson-Darling Goodness of Fit
Chi-Square Goodness of Fit
Kolmogorov-Smirnov Goodness of Fit

Case Study

The probability plot is demonstrated in the uniform random numbers case study.

Software

Most general purpose statistical software programs support probability plots for at least a few common distributions.

1.3.3.23. Probability Plot Correlation Coefficient Plot

Purpose: Graphical Technique for Finding the Shape Parameter of a Distributional Family that Best Fits a Data Set

The probability plot correlation coefficient (PPCC) plot (Filliben 1975) is a graphical technique for identifying the shape parameter for a distributional family that best describes the data set. This technique is appropriate for families, such as the Weibull, that are defined by a single shape parameter and location and scale parameters, and it is not appropriate for distributions, such as the normal, that are defined only by location and scale parameters.

The PPCC plot is generated as follows. For a series of values for the shape parameter, the correlation coefficient is computed for the probability plot associated with a given value of the shape parameter. These correlation coefficients are plotted against their corresponding shape parameters. The maximum correlation coefficient corresponds to the optimal value of the shape parameter. For better precision, two iterations of the PPCC plot can be generated; the first is for finding the right neighborhood and the second is for fine tuning the estimate.

The PPCC plot is used first to find a good value of the shape parameter. The probability plot is then generated to find estimates of the location and scale parameters and in addition to provide a graphical assessment of the adequacy of the distributional fit.

Compare Distributions

In addition to finding a good choice for estimating the shape parameter of a given distribution, the PPCC plot can be useful in deciding which distributional family is most appropriate. For example, given a set of reliabilty data, you might generate PPCC plots for a Weibull, lognormal, gamma, and inverse Gaussian distributions, and possibly others, on a single page. This one page would show the best value for the shape parameter for several distributions and would additionally indicate which of these distributional families provides the best fit (as measured by the maximum probability plot correlation coefficient). That is, if the maximum PPCC value for the Weibull is 0.99 and only 0.94 for the lognormal, then we could reasonably conclude that the Weibull family is the better choice.

Tukey-Lambda PPCC Plot for Symmetric Distributions

The Tukey Lambda PPCC plot, with shape parameter λ, is particularly useful for symmetric distributions. It indicates whether a distribution is short or long tailed and it can further indicate several common distributions. Specifically,

1. λ = -1: distribution is approximately Cauchy
2. λ = 0: distribution is exactly logistic
3. λ = 0.14: distribution is approximately normal
4. λ = 0.5: distribution is U-shaped
5. λ = 1: distribution is exactly uniform

If the Tukey Lambda PPCC plot gives a maximum value near 0.14, we can reasonably conclude that the normal distribution is a good model for the data. If the maximum value is less than 0.14, a long-tailed distribution such as the double exponential or logistic would be a better choice. If the maximum value is near -1, this implies the selection of very long-tailed distribution, such as the Cauchy. If the maximum value is greater than 0.14, this implies a short-tailed distribution such as the Beta or uniform.

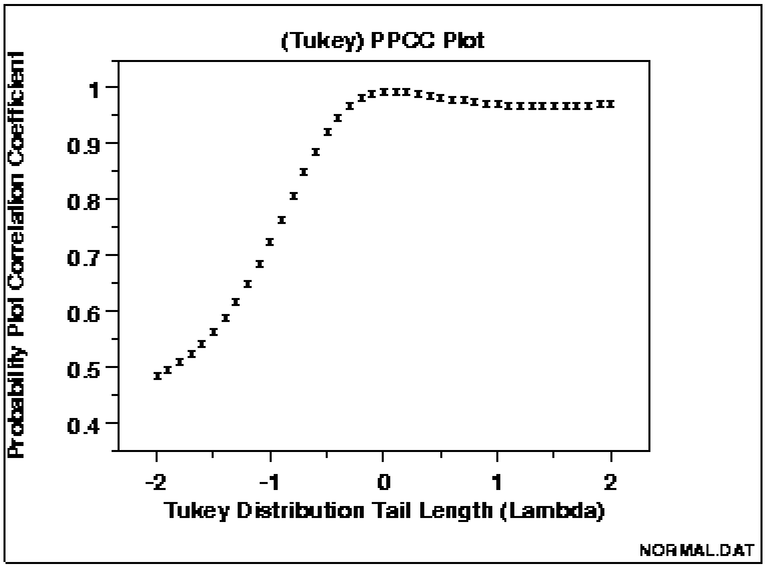
The Tukey-Lambda PPCC plot is used to suggest an appropriate distribution. You should follow-up with PPCC and probability plots of the appropriate alternatives.

Use Judgement When Selecting An Appropriate Distributional Family

When comparing distributional models, do not simply choose the one with the maximum PPCC value. In many cases, several distributional fits provide comparable PPCC values. For example, a lognormal and Weibull may both fit a given set of reliability data quite well. Typically, we would consider the complexity of the distribution. That is, a simpler distribution with a marginally smaller PPCC value may be preferred over a more complex distribution. Likewise, there may be theoretical justification in terms of the underlying scientific model for preferring a distribution with a marginally smaller PPCC value in some cases. In other cases, we may not need to know if the distributional model is optimal, only that it is adequate for our purposes. That is, we may be able to use techniques designed for normally distributed data even if other distributions fit the data somewhat better.

Sample Plot

The following is a PPCC plot of 100 normal random numbers. The maximum value of the correlation coefficient = 0.997 at λ = 0.099.



This PPCC plot shows that:

1. the best-fit symmetric distribution is nearly normal;
2. the data are not long tailed;
3. the sample mean would be an appropriate estimator of location.

We can follow-up this PPCC plot with a normal probability plot to verify the normality model for the data.

Definition:

The PPCC plot is formed by:

- Vertical axis: Probability plot correlation coefficient;
- Horizontal axis: Value of shape parameter.

Questions

The PPCC plot answers the following questions:

1. What is the best-fit member within a distributional family?
2. Does the best-fit member provide a good fit (in terms of generating a probability plot with a high correlation coefficient)?
3. Does this distributional family provide a good fit compared to other distributions?
4. How sensitive is the choice of the shape parameter?

Importance

Many statistical analyses are based on distributional assumptions about the population from which the data have been obtained. However, distributional families can have radically different shapes depending on the value of the shape parameter. Therefore, finding a reasonable choice for the shape parameter is a necessary step in the analysis. In many analyses, finding a good distributional model for the data is the primary focus of the analysis. In both of these cases, the PPCC plot is a valuable tool.

Related Techniques	Probability Plot Maximum Likelihood Estimation Least Squares Estimation Method of Moments Estimation
Software	PPCC plots are currently not available in most common general purpose statistical software programs. However, the underlying technique is based on probability plots and correlation coefficients, so it should be possible to write macros for PPCC plots in statistical programs that support these capabilities. Dataplot supports PPCC plots.

1.3.3.24. Quantile-Quantile Plot

Purpose: Check If Two Data Sets Can Be Fit With the Same Distribution

The quantile-quantile (q-q) plot is a graphical technique for determining if two data sets come from populations with a common distribution.

A q-q plot is a plot of the quantiles of the first data set against the quantiles of the second data set. By a quantile, we mean the fraction (or percent) of points below the given value. That is, the 0.3 (or 30%) quantile is the point at which 30% percent of the data fall below and 70% fall above that value.

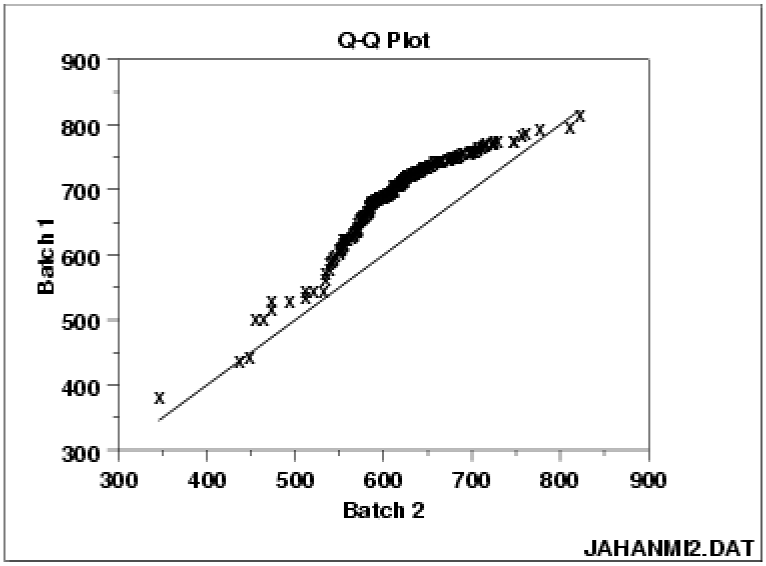
A 45-degree reference line is also plotted. If the two sets come from a population with the same distribution, the points should fall approximately along this reference line. The greater the departure from this reference line, the greater the evidence for the conclusion that the two data sets have come from populations with different distributions.

The advantages of the q-q plot are:

1. The sample sizes do not need to be equal.
2. Many distributional aspects can be simultaneously tested. For example, shifts in location, shifts in scale, changes in symmetry, and the presence of outliers can all be detected from this plot. For example, if the two data sets come from populations whose distributions differ only by a shift in location, the points should lie along a straight line that is displaced either up or down from the 45-degree reference line.

The q-q plot is similar to a probability plot. For a probability plot, the quantiles for one of the data samples are replaced with the quantiles of a theoretical distribution.

Sample Plot



This q-q plot shows that

1. These 2 batches do not appear to have come from populations with a common distribution.
2. The batch 1 values are significantly higher than the corresponding batch 2 values.
3. The differences are increasing from values 525 to 625. Then the values for the 2 batches get closer again.

Definition: Quantiles for Data Set 1 Versus Quantiles of Data Set 2

The q-q plot is formed by:

- Vertical axis: Estimated quantiles from data set 1
- Horizontal axis: Estimated quantiles from data set 2

Both axes are in units of their respective data sets. That is, the actual quantile level is not plotted. For a given point on the q-q plot, we know that the quantile level is the same for both points, but not what that quantile level actually is.

If the data sets have the same size, the q-q plot is essentially a plot of sorted data set 1 against sorted data set 2. If the data sets are not of equal size, the quantiles are usually picked to correspond to the sorted values from the smaller data set and then the quantiles for the larger data set are interpolated.

Questions

The q-q plot is used to answer the following questions:

- Do two data sets come from populations with a common distribution?
- Do two data sets have common location and scale?
- Do two data sets have similar distributional shapes?
- Do two data sets have similar tail behavior?

Importance: Check for Common

When there are two data samples, it is often desirable to know if the assumption of a common distribution is justified. If so, then location and scale estimators can pool both data

Distribution	sets to obtain estimates of the common location and scale. If two samples do differ, it is also useful to gain some understanding of the differences. The q-q plot can provide more insight into the nature of the difference than analytical methods such as the chi-square and Kolmogorov-Smirnov 2-sample tests.
Related Techniques	Bihistogram T Test F Test 2-Sample Chi-Square Test 2-Sample Kolmogorov-Smirnov Test
Case Study	The quantile-quantile plot is demonstrated in the ceramic strength data case study.
Software	Q-Q plots are available in some general purpose statistical software programs. If the number of data points in the two samples are equal, it should be relatively easy to write a macro in statistical programs that do not support the q-q plot. If the number of points are not equal, writing a macro for a q-q plot may be difficult.

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.3. Graphical Techniques: Alphabetic

1.3.3.25. Run-Sequence Plot

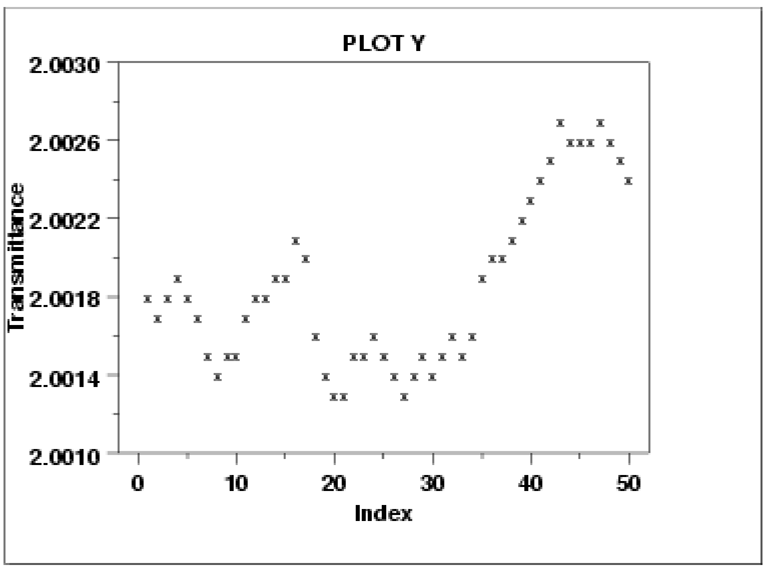
Purpose: Check for Shifts in Location and Scale and Outliers

Run sequence plots (Chambers 1983) are an easy way to graphically summarize a univariate data set. A common assumption of univariate data sets is that they behave like:

1. random drawings;
2. from a fixed distribution;
3. with a common location; and
4. with a common scale.

With run sequence plots, shifts in location and scale are typically quite evident. Also, outliers can easily be detected.

Sample Plot: Last Third of Data Shows a Shift of Location



This sample run sequence plot shows that the location shifts up for the last third of the data.

Definition: y(i) Versus i

Run sequence plots are formed by:

- Vertical axis: Response variable Y(i)
- Horizontal axis: Index i (i = 1, 2, 3, ...)

Questions

The run sequence plot can be used to answer the following questions

1. Are there any shifts in location?
2. Are there any shifts in variation?
3. Are there any outliers?

The run sequence plot can also give the analyst an excellent feel for the data.

Importance: Check Univariate Assumptions

For univariate data, the default model is

Y = constant + error

where the error is assumed to be random, from a fixed distribution, and with constant location and scale. The validity of this model depends on the validity of these assumptions. The run sequence plot is useful for checking for constant location and scale.

Even for more complex models, the assumptions on the error term are still often the same. That is, a run sequence plot of the residuals (even from very complex models) is still vital for checking for outliers and for detecting shifts in location and scale.

Related Techniques

Scatter Plot
Histogram
Autocorrelation Plot
Lag Plot

Case Study

The run sequence plot is demonstrated in the Filter transmittance data case study.

Software

Run sequence plots are available in most general purpose statistical software programs.

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

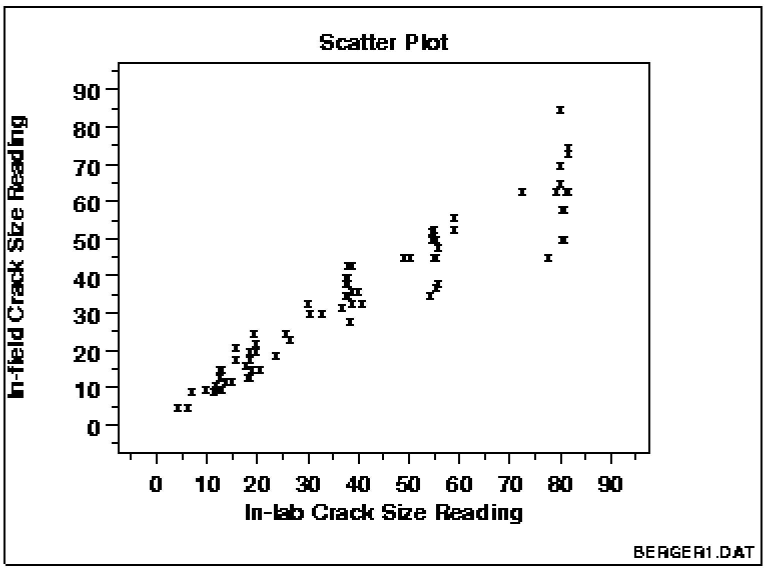
1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.3. Graphical Techniques: Alphabetic

1.3.3.26. Scatter Plot

Purpose: Check for Relationship

A scatter plot (Chambers 1983) reveals relationships or association between two variables. Such relationships manifest themselves by any non-random structure in the plot. Various common types of patterns are demonstrated in the examples.

Sample Plot: Linear Relationship Between Variables Y and X



This sample plot reveals a linear relationship between the two variables indicating that a linear regression model might be appropriate.

Definition: Y Versus X

A scatter plot is a plot of the values of Y versus the corresponding values of X:

- Vertical axis: variable Y--usually the response variable
- Horizontal axis: variable X--usually some variable we suspect may ber related to the response

Questions

Scatter plots can provide answers to the following questions:

1. Are variables X and Y related?
2. Are variables X and Y linearly related?
3. Are variables X and Y non-linearly related?
4. Does the variation in Y change depending on X?
5. Are there outliers?

Examples

1. No relationship
2. Strong linear (positive correlation)
3. Strong linear (negative correlation)
4. Exact linear (positive correlation)
5. Quadratic relationship
6. Exponential relationship
7. Sinusoidal relationship (damped)
8. Variation of Y doesn't depend on X (homoscedastic)
9. Variation of Y does depend on X (heteroscedastic)
10. Outlier

Combining Scatter Plots

Scatter plots can also be combined in multiple plots per page to help understand higher-level structure in data sets with more than two variables.

The scatterplot matrix generates all pairwise scatter plots on a single page. The conditioning plot, also called a co-plot or subset plot, generates scatter plots of Y versus X dependent on the value of a third variable.

Causality Is Not Proved By Association

The scatter plot uncovers relationships in data. "Relationships" means that there is some structured association (linear, quadratic, etc.) between X and Y. Note, however, that even though

causality implies association

association does NOT imply causality.

Scatter plots are a useful diagnostic tool for determining association, but if such association exists, the plot may or may not suggest an underlying cause-and-effect mechanism. A scatter plot can never "prove" cause and effect--it is ultimately only the researcher (relying on the underlying science/engineering) who can conclude that causality actually exists.

Appearance

The most popular rendition of a scatter plot is

1. some plot character (e.g., X) at the data points, and
2. no line connecting data points.

Other scatter plot format variants include

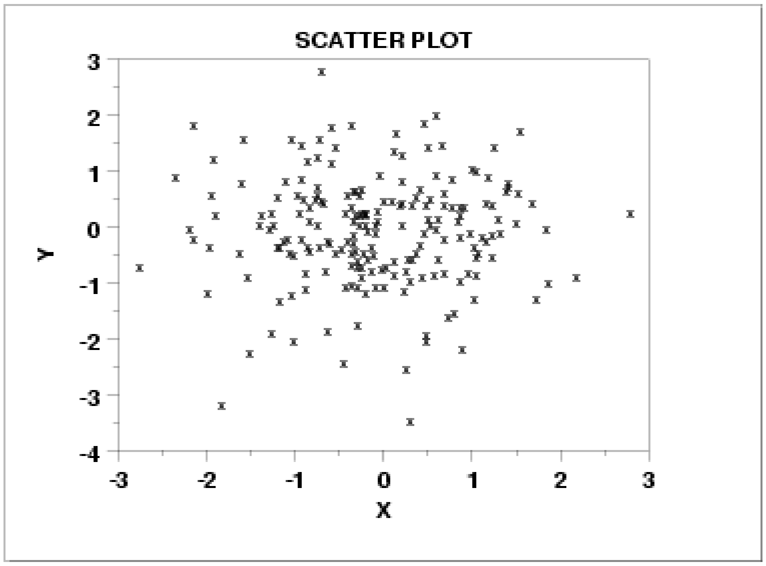
1. an optional plot character (e.g, X) at the data points, but
2. a solid line connecting data points.

In both cases, the resulting plot is referred to as a scatter plot, although the former (discrete and disconnected) is the author's personal preference since nothing makes it onto the screen except the data--there are no interpolative artifacts to bias the interpretation.

Related Techniques	Run Sequence Plot Box Plot Block Plot
Case Study	The scatter plot is demonstrated in the load cell calibration data case study.
Software	Scatter plots are a fundamental technique that should be available in any general purpose statistical software program. Scatter plots are also available in most graphics and spreadsheet programs as well.

1.3.3.26.1. Scatter Plot: No Relationship

Scatter Plot with No Relationship

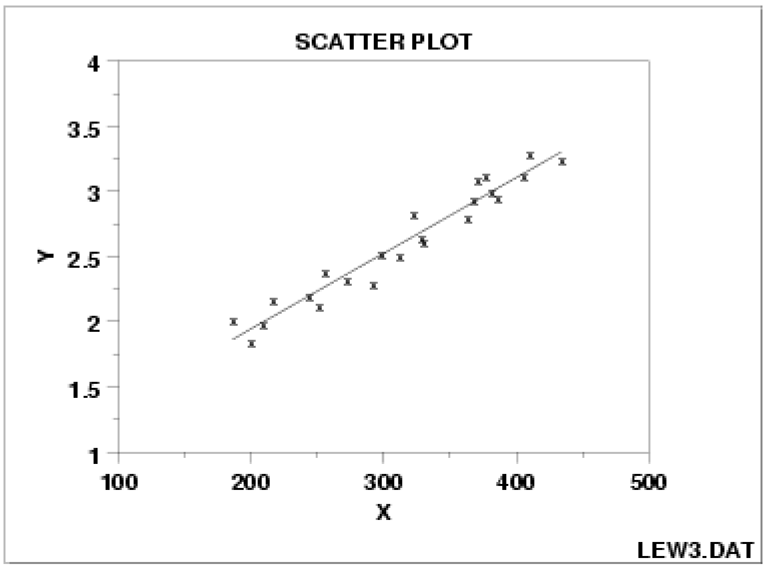


Discussion

Note in the plot above how for a given value of *X* (say *X* = 0.5), the corresponding values of *Y* range all over the place from *Y* = -2 to *Y* = +2. The same is true for other values of *X*. This lack of predictablility in determining *Y* from a given value of *X*, and the associated amorphous, non-structured appearance of the scatter plot leads to the summary conclusion: no relationship.

1.3.3.26.2. Scatter Plot: Strong Linear (positive correlation) Relationship

Scatter Plot Showing Strong Positive Linear Correlation

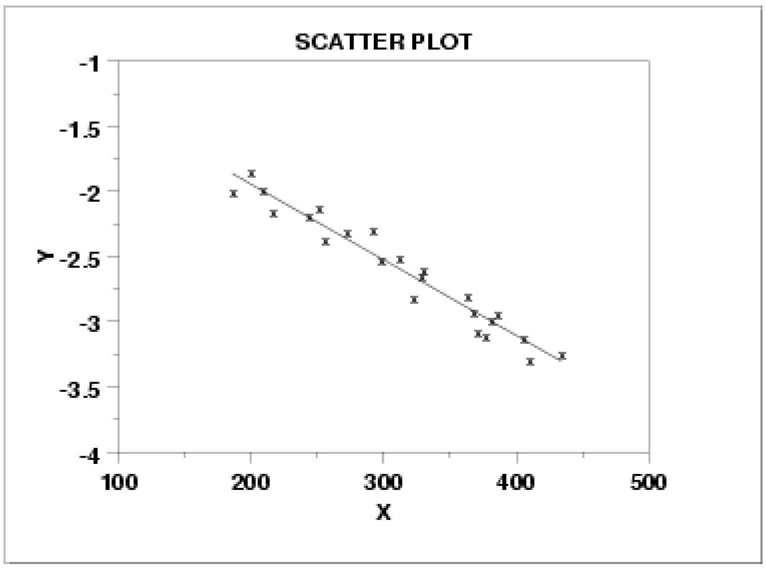


Discussion

Note in the plot above how a straight line comfortably fits through the data; hence a linear relationship exists. The scatter about the line is quite small, so there is a strong linear relationship. The slope of the line is positive (small values of *X* correspond to small values of *Y*; large values of *X* correspond to large values of *Y*), so there is a positive co-relation (that is, a positive correlation) between *X* and *Y*.

1.3.3.26.3. Scatter Plot: Strong Linear (negative correlation) Relationship

Scatter Plot Showing a Strong Negative Correlation

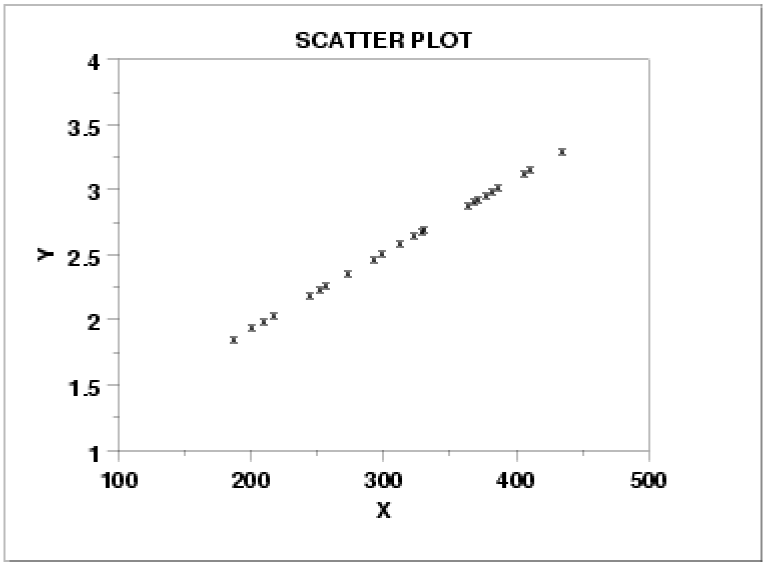


Discussion

Note in the plot above how a straight line comfortably fits through the data; hence there is a linear relationship. The scatter about the line is quite small, so there is a strong linear relationship. The slope of the line is negative (small values of *X* correspond to large values of *Y*; large values of *X* correspond to small values of *Y*), so there is a negative co-relation (that is, a negative correlation) between *X* and *Y*.

1.3.3.26.4. Scatter Plot: Exact Linear (positive correlation) Relationship

Scatter Plot Showing an Exact Linear Relationship

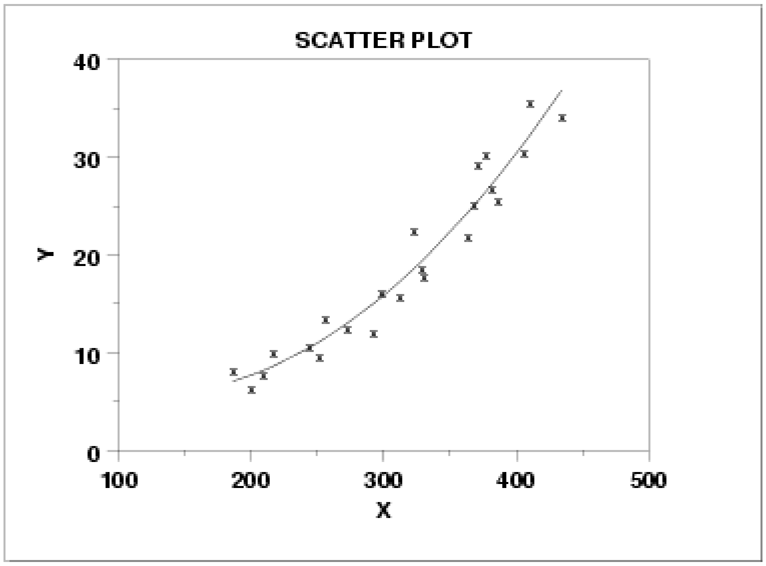


Discussion

Note in the plot above how a straight line comfortably fits through the data; hence there is a linear relationship. The scatter about the line is zero--there is perfect predictability between *X* and *Y*), so there is an exact linear relationship. The slope of the line is positive (small values of *X* correspond to small values of *Y*; large values of *X* correspond to large values of *Y*), so there is a positive co-relation (that is, a positive correlation) between *X* and *Y*.

1.3.3.26.5. Scatter Plot: Quadratic Relationship

Scatter Plot Showing Quadratic Relationship



Discussion

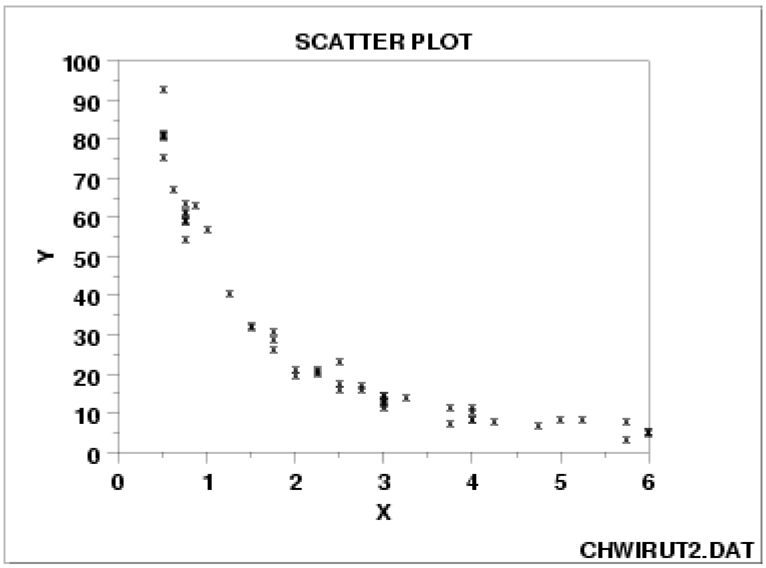
Note in the plot above how no imaginable simple straight line could ever adequately describe the relationship between *X* and *Y*--a curved (or curvilinear, or non-linear) function is needed. The simplest such curvilinear function is a quadratic model

$$Y_i = A + BX_i + CX_i^2 + E_i$$

for some A, B, and C. Many other curvilinear functions are possible, but the data analysis principle of parsimony suggests that we try fitting a quadratic function first.

1.3.3.26.6. Scatter Plot: Exponential Relationship

Scatter Plot Showing Exponential Relationship



Discussion

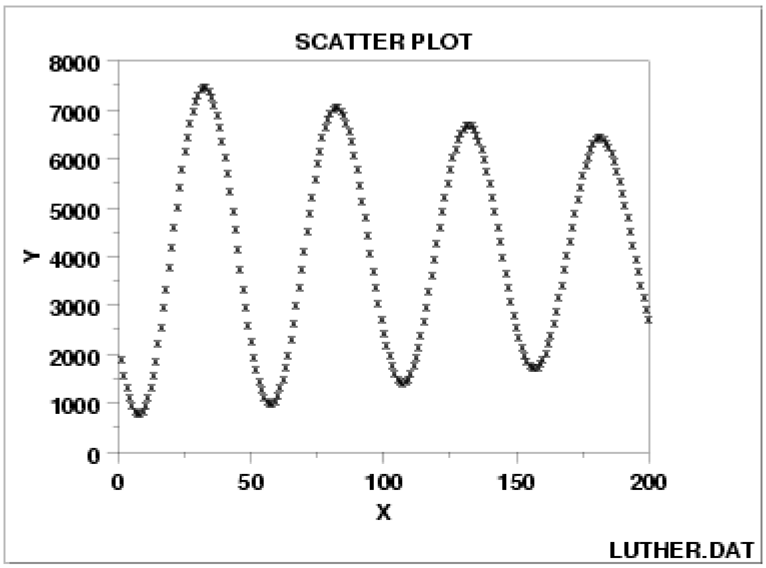
Note that a simple straight line is grossly inadequate in describing the relationship between *X* and *Y*. A quadratic model would prove lacking, especially for large values of *X*. In this example, the large values of *X* correspond to nearly constant values of *Y*, and so a non-linear function beyond the quadratic is needed. Among the many other non-linear functions available, one of the simpler ones is the exponential model

$$Y_i = A + Be^{CX_i} + E_i$$

for some A, B, and C. In this case, an exponential function would, in fact, fit well, and so one is led to the summary conclusion of an exponential relationship.

1.3.3.26.7. Scatter Plot: Sinusoidal Relationship (damped)

Scatter Plot Showing a Sinusoidal Relationship



Discussion

The complex relationship between *X* and *Y* appears to be basically oscillatory, and so one is naturally drawn to the trigonometric sinusoidal model:

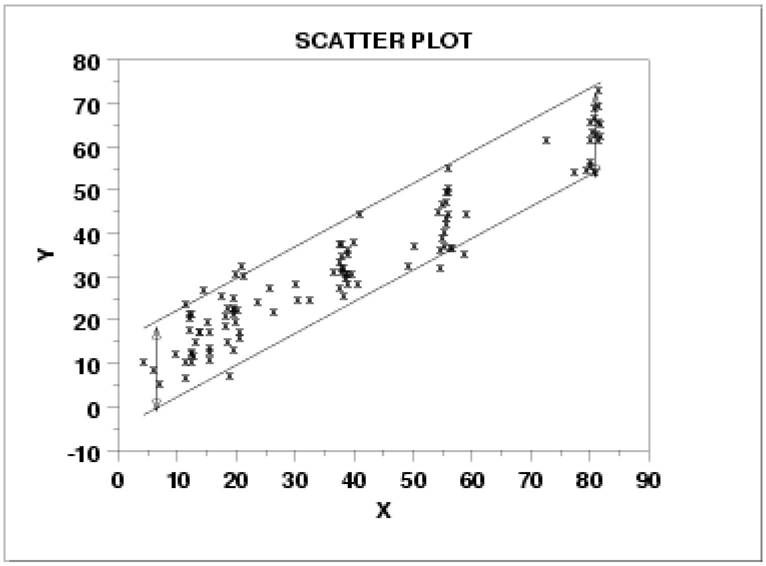
$$Y_i = C + \alpha \sin\left(2\pi\omega t_i + \phi\right) + E_i$$

Closer inspection of the scatter plot reveals that the amount of swing (the amplitude α in the model) does not appear to be constant but rather is decreasing (damping) as *X* gets large. We thus would be led to the conclusion: damped sinusoidal relationship, with the simplest corresponding model being

$$Y_i = C + \left(B_0 + B_1 * t_i\right) \sin\left(2\pi\omega t_i + \phi\right) + E_i$$

1.3.3.26.8. Scatter Plot: Variation of Y Does Not Depend on X (homoscedastic)

Scatter Plot Showing Homoscedastic Variability

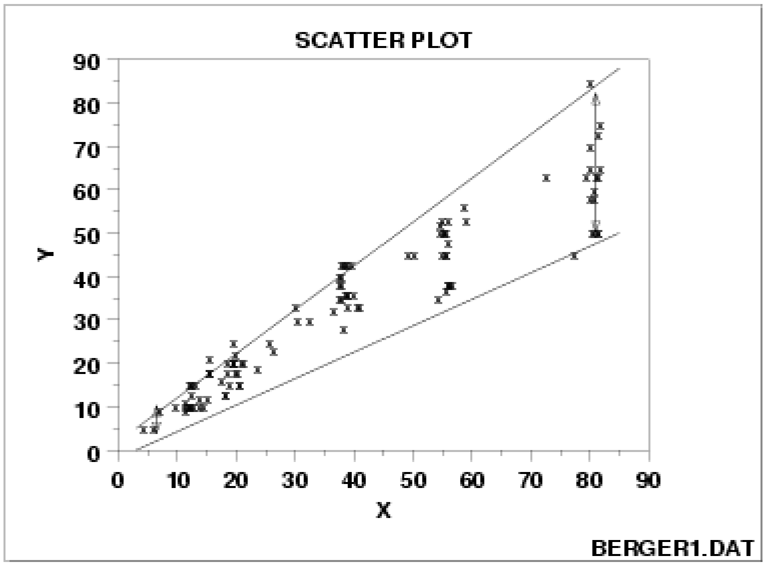


Discussion

This scatter plot reveals a linear relationship between *X* and *Y*: for a given value of *X*, the predicted value of *Y* will fall on a line. The plot further reveals that the variation in *Y* about the predicted value is about the same (+- 10 units), regardless of the value of *X*. Statistically, this is referred to as homoscedasticity. Such homoscedasticity is very important as it is an underlying assumption for regression, and its violation leads to parameter estimates with inflated variances. If the data are homoscedastic, then the usual regression estimates can be used. If the data are not homoscedastic, then the estimates can be improved using weighting procedures as shown in the next example.

1.3.3.26.9. Scatter Plot: Variation of Y Does Depend on X (heteroscedastic)

Scatter Plot Showing Heteroscedastic Variability



Discussion

This scatter plot reveals an approximate linear relationship between *X* and *Y*, but more importantly, it reveals a statistical condition referred to as heteroscedasticity (that is, nonconstant variation in *Y* over the values of *X*). For a heteroscedastic data set, the variation in *Y* differs depending on the value of *X*. In this example, small values of *X* yield small scatter in *Y* while large values of *X* result in large scatter in *Y*.

Heteroscedasticity complicates the analysis somewhat, but its effects can be overcome by:

1. proper weighting of the data with noisier data being weighted less, or by

2. performing a *Y* variable transformation to achieve homoscedasticity. The Box-Cox normality plot can help determine a suitable transformation.

Impact of Ignoring Unequal

Fortunately, unweighted regression analyses on heteroscedastic data produce estimates of the coefficients that are unbiased. However, the coefficients will not be as

Variability in the Data

precise as they would be with proper weighting.

Note further that if heteroscedasticity does exist, it is frequently useful to plot and model the local variation $var(Y_i|X_i)$ as a function of *X*, as in $var(Y_i|X_i) = g(X_i)$. This modeling has two advantages:

1. it provides additional insight and understanding as to how the response *Y* relates to *X*; and

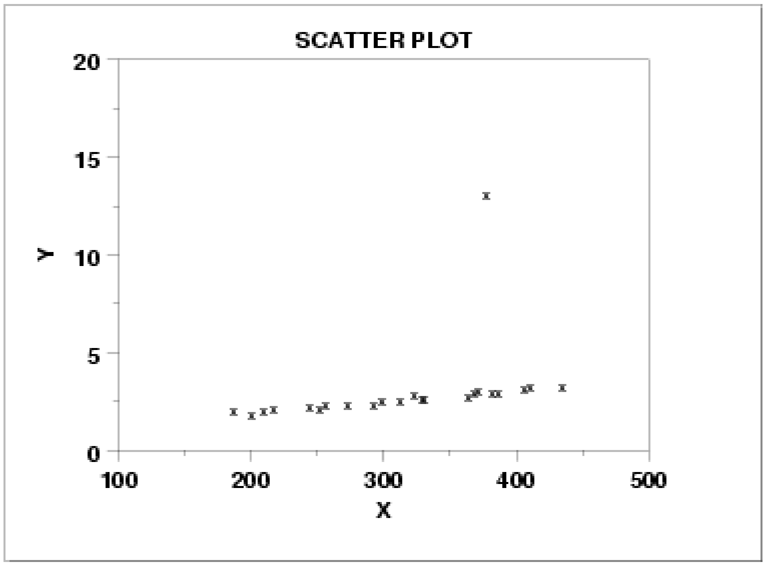
2. it provides a convenient means of forming weights for a weighted regression by simply using

$$w_i = W(Y_i|X_i) = \frac{1}{Var(Y_i|X_i)} = \frac{1}{g(X_i)}$$

The topic of non-constant variation is discussed in some detail in the process modeling chapter.

1.3.3.26.10. Scatter Plot: Outlier

Scatter Plot Showing Outliers



Discussion

The scatter plot here reveals

1. a basic linear relationship between *X* and *Y* for most of the data, and
2. a single outlier (at *X* = 375).

An outlier is defined as a data point that emanates from a different model than do the rest of the data. The data here appear to come from a linear model with a given slope and variation except for the outlier which appears to have been generated from some other model.

Outlier detection is important for effective modeling. Outliers should be excluded from such model fitting. If all the data here are included in a linear regression, then the fitted model will be poor virtually everywhere. If the outlier is omitted from the fitting process, then the resulting fit will be excellent almost everywhere (for all points except the outlying point).

1.3.3.26.11. Scatterplot Matrix

Purpose: Check Pairwise Relationships Between Variables

Given a set of variables $X_1, X_2, \ldots, X_k$, the scatterplot matrix contains all the pairwise scatter plots of the variables on a single page in a matrix format. That is, if there are *k* variables, the scatterplot matrix will have *k* rows and *k* columns and the *i*th row and *j*th column of this matrix is a plot of X_i versus X_j.

Although the basic concept of the scatterplot matrix is simple, there are numerous alternatives in the details of the plots.

1. The diagonal plot is simply a 45-degree line since we are plotting X_i versus X_i. Although this has some usefulness in terms of showing the univariate distribution of the variable, other alternatives are common. Some users prefer to use the diagonal to print the variable label. Another alternative is to plot the univariate histogram on the diagonal. Alternatively, we could simply leave the diagonal blank.

2. Since X_i versus X_j is equivalent to X_j versus X_i with the axes reversed, some prefer to omit the plots below the diagonal.

3. It can be helpful to overlay some type of fitted curve on the scatter plot. Although a linear or quadratic fit can be used, the most common alternative is to overlay a lowess curve.

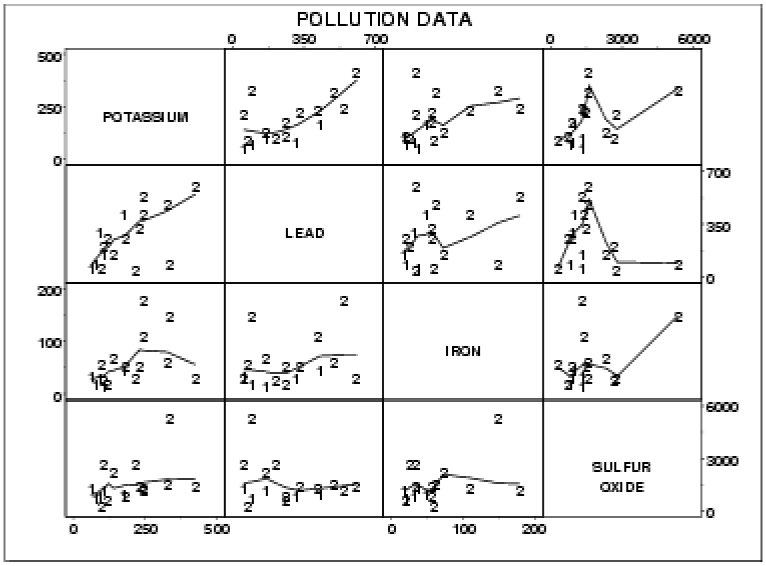
4. Due to the potentially large number of plots, it can be somewhat tricky to provide the axes labels in a way that is both informative and visually pleasing. One alternative that seems to work well is to provide axis labels on alternating rows and columns. That is, row one will have tic marks and axis labels on the left vertical axis for the first plot only while row two will have the tic marks and axis labels for the right vertical axis for the last plot in the row only. This alternating pattern continues for the remaining rows.

A similar pattern is used for the columns and the horizontal axes labels. Another alternative is to put the minimum and maximum scale value in the diagonal plot with the variable name.

5. Some analysts prefer to connect the scatter plots. Others prefer to leave a little gap between each plot.

6. Although this plot type is most commonly used for scatter plots, the basic concept is both simple and powerful and extends easily to other plot formats that involve pairwise plots such as the quantile-quantile plot and the bihistogram.

Sample Plot



This sample plot was generated from pollution data collected by NIST chemist Lloyd Currie.

There are a number of ways to view this plot. If we are primarily interested in a particular variable, we can scan the row and column for that variable. If we are interested in finding the strongest relationship, we can scan all the plots and then determine which variables are related.

Definition

Given *k* variables, scatter plot matrices are formed by creating *k* rows and *k* columns. Each row and column defines a single scatter plot

The individual plot for row *i* and column *j* is defined as

- Vertical axis: Variable X_i
- Horizontal axis: Variable X_j

Questions

The scatterplot matrix can provide answers to the following questions:

1. Are there pairwise relationships between the variables?
2. If there are relationships, what is the nature of these relationships?
3. Are there outliers in the data?
4. Is there clustering by groups in the data?

Linking and Brushing

The scatterplot matrix serves as the foundation for the concepts of linking and brushing.

By linking, we mean showing how a point, or set of points, behaves in each of the plots. This is accomplished by highlighting these points in some fashion. For example, the highlighted points could be drawn as a filled circle while the remaining points could be drawn as unfilled circles. A typical application of this would be to show how an outlier shows up in each of the individual pairwise plots. Brushing extends this concept a bit further. In brushing, the points to be highlighted are interactively selected by a mouse and the scatterplot matrix is dynamically updated (ideally in real time). That is, we can select a rectangular region of points in one plot and see how those points are reflected in the other plots. Brushing is discussed in detail by Becker, Cleveland, and Wilks in the paper *"Dynamic Graphics for Data Analysis"* (Cleveland and McGill, 1988).

Related Techniques

Star plot
Scatter plot
Conditioning plot
Locally weighted least squares

Software

Scatterplot matrices are becoming increasingly common in general purpose statistical software programs. If a software program does not generate scatterplot matrices, but it does provide multiple plots per page and scatter plots, it should be possible to write a macro to generate a scatterplot matrix. Brushing is available in a few of the general purpose statistical software programs that emphasize graphical approaches.

1.3.3.26.12. Conditioning Plot

Purpose: Check pairwise relationship between two variables conditional on a third variable

A conditioning plot, also known as a coplot or subset plot, is a plot of two variables conditional on the value of a third variable (called the conditioning variable). The conditioning variable may be either a variable that takes on only a few discrete values or a continuous variable that is divided into a limited number of subsets.

One limitation of the scatterplot matrix is that it cannot show interaction effects with another variable. This is the strength of the conditioning plot. It is also useful for displaying scatter plots for groups in the data. Although these groups can also be plotted on a single plot with different plot symbols, it can often be visually easier to distinguish the groups using the conditioning plot.

Although the basic concept of the conditioning plot matrix is simple, there are numerous alternatives in the details of the plots.

1. It can be helpful to overlay some type of fitted curve on the scatter plot. Although a linear or quadratic fit can be used, the most common alternative is to overlay a lowess curve.

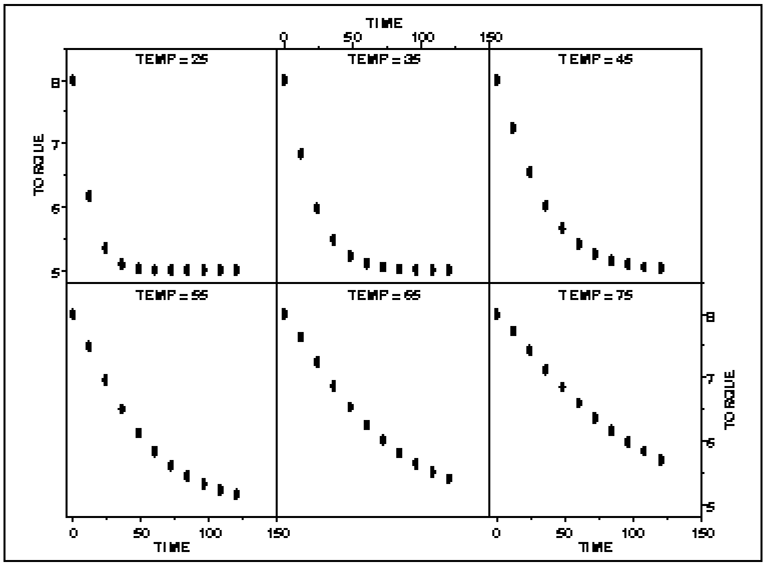
2. Due to the potentially large number of plots, it can be somewhat tricky to provide the axis labels in a way that is both informative and visually pleasing. One alternative that seems to work well is to provide axis labels on alternating rows and columns. That is, row one will have tic marks and axis labels on the left vertical axis for the first plot only while row two will have the tic marks and axis labels for the right vertical axis for the last plot in the row only. This alternating pattern continues for the remaining rows. A similar pattern is used for the columns and the horizontal axis labels. Note that this approach only works if the axes limits are fixed to common values for all of the plots.

3. Some analysts prefer to connect the scatter plots. Others prefer to leave a little gap between each plot. Alternatively, each plot can have its own labeling with

the plots not connected.

4. Although this plot type is most commonly used for scatter plots, the basic concept is both simple and powerful and extends easily to other plot formats.

Sample Plot



In this case, temperature has six distinct values. We plot torque versus time for each of these temperatures. This example is discussed in more detail in the process modeling chapter.

Definition

Given the variables ***X***, ***Y***, and ***Z***, the conditioning plot is formed by dividing the values of ***Z*** into *k* groups. There are several ways that these groups may be formed. There may be a natural grouping of the data, the data may be divided into several equal sized groups, the grouping may be determined by clusters in the data, and so on. The page will be divided into *n* rows and *c* columns where $nc \geq k$. Each row and column defines a single scatter plot.

The individual plot for row *i* and column *j* is defined as

- Vertical axis: Variable ***Y***
- Horizontal axis: Variable ***X***

where only the points in the group corresponding to the *i*th row and *j*th column are used.

Questions

The conditioning plot can provide answers to the following questions:

1. Is there a relationship between two variables?
2. If there is a relationship, does the nature of the relationship depend on the value of a third variable?
3. Are groups in the data similar?
4. Are there outliers in the data?

Related Techniques	Scatter plot Scatterplot matrix Locally weighted least squares
Software	Scatter plot matrices are becoming increasingly common in general purpose statistical software programs, including. If a software program does not generate conditioning plots, but it does provide multiple plots per page and scatter plots, it should be possible to write a macro to generate a conditioning plot.

1.3.3.27. Spectral Plot

Purpose: Examine Cyclic Structure

A spectral plot (Jenkins and Watts 1968 or Bloomfield 1976) is a graphical technique for examining cyclic structure in the frequency domain. It is a smoothed Fourier transform of the autocovariance function.

The frequency is measured in cycles per unit time where unit time is defined to be the distance between 2 points. A frequency of 0 corresponds to an infinite cycle while a frequency of 0.5 corresponds to a cycle of 2 data points. Equi-spaced time series are inherently limited to detecting frequencies between 0 and 0.5.

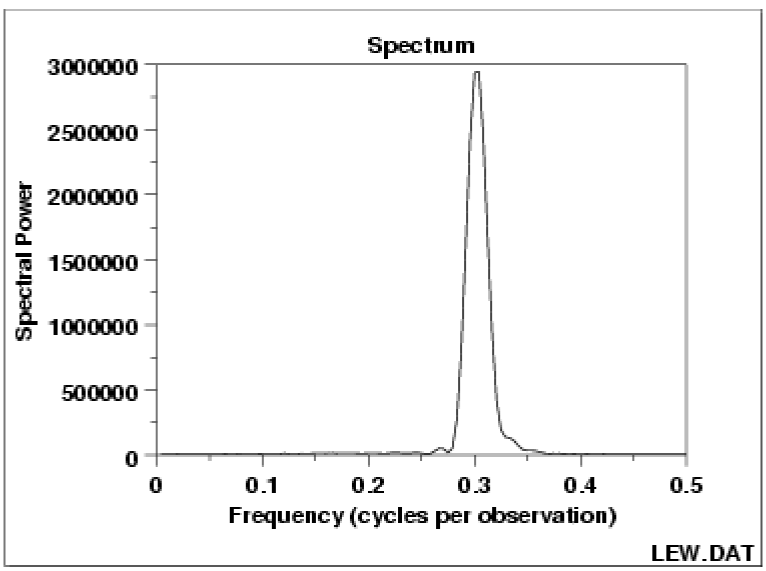
Trends should typically be removed from the time series before applying the spectral plot. Trends can be detected from a run sequence plot. Trends are typically removed by differencing the series or by fitting a straight line (or some other polynomial curve) and applying the spectral analysis to the residuals.

Spectral plots are often used to find a starting value for the frequency, ω, in the sinusoidal model

$$Y_i = C + \alpha \sin(2\pi\omega t_i + \phi) + E_i$$

See the beam deflection case study for an example of this.

Sample Plot



This spectral plot shows one dominant frequency of approximately 0.3 cycles per observation.

Definition: Variance Versus Frequency

The spectral plot is formed by:

- Vertical axis: Smoothed variance (power)
- Horizontal axis: Frequency (cycles per observation)

The computations for generating the smoothed variances can be involved and are not discussed further here. The details can be found in the Jenkins and Bloomfield references and in most texts that discuss the frequency analysis of time series.

Questions

The spectral plot can be used to answer the following questions:

1. How many cyclic components are there?
2. Is there a dominant cyclic frequency?
3. If there is a dominant cyclic frequency, what is it?

Importance Check Cyclic Behavior of Time Series

The spectral plot is the primary technique for assessing the cyclic nature of univariate time series in the frequency domain. It is almost always the second plot (after a run sequence plot) generated in a frequency domain analysis of a time series.

Examples

1. Random (= White Noise)
2. Strong autocorrelation and autoregressive model
3. Sinusoidal model

Related Techniques

Autocorrelation Plot
Complex Demodulation Amplitude Plot
Complex Demodulation Phase Plot

Case Study

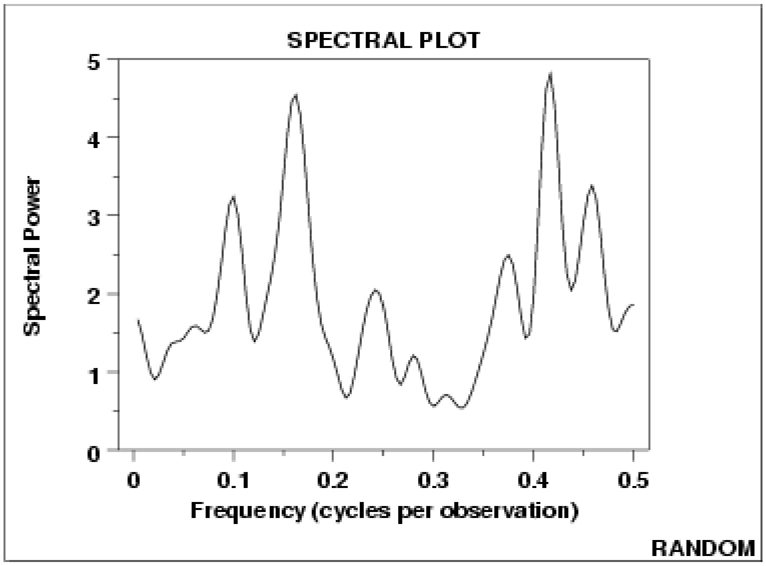
The spectral plot is demonstrated in the beam deflection data case study.

Software

Spectral plots are a fundamental technique in the frequency analysis of time series. They are available in many general purpose statistical software programs.

1.3.3.27.1. Spectral Plot: Random Data

Spectral Plot of 200 Normal Random Numbers



Conclusions

We can make the following conclusions from the above plot.

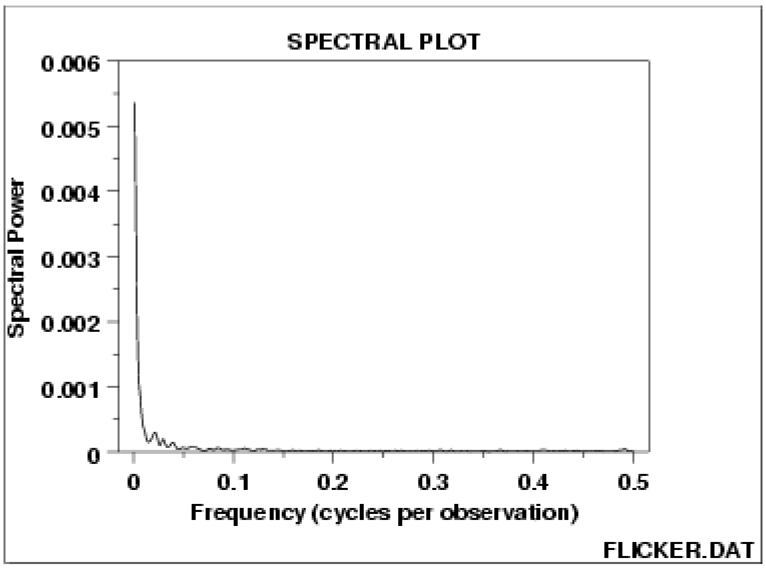
1. There are no dominant peaks.
2. There is no identifiable pattern in the spectrum.
3. The data are random.

Discussion

For random data, the spectral plot should show no dominant peaks or distinct pattern in the spectrum. For the sample plot above, there are no clearly dominant peaks and the peaks seem to fluctuate at random. This type of appearance of the spectral plot indicates that there are no significant cyclic patterns in the data.

1.3.3.27.2. Spectral Plot: Strong Autocorrelation and Autoregressive Model

Spectral Plot for Random Walk Data



Conclusions

We can make the following conclusions from the above plot.

1. Strong dominant peak near zero.
2. Peak decays rapidly towards zero.
3. An autoregressive model is an appropriate model.

Discussion

This spectral plot starts with a dominant peak near zero and rapidly decays to zero. This is the spectral plot signature of a process with strong positive autocorrelation. Such processes are highly non-random in that there is high association between an observation and a succeeding observation. In short, if you know Y_i you can make a strong guess as to what Y_{i+1} will be.

Recommended Next Step

The next step would be to determine the parameters for the autoregressive model:

$$Y_i = A_0 + A_1 * Y_{i-1} + E_i$$

Such estimation can be done by linear regression or by fitting a Box-Jenkins autoregressive (AR) model.

The residual standard deviation for this autoregressive model will be much smaller than the residual standard deviation for the default model

$$Y_i = A_0 + E_i$$

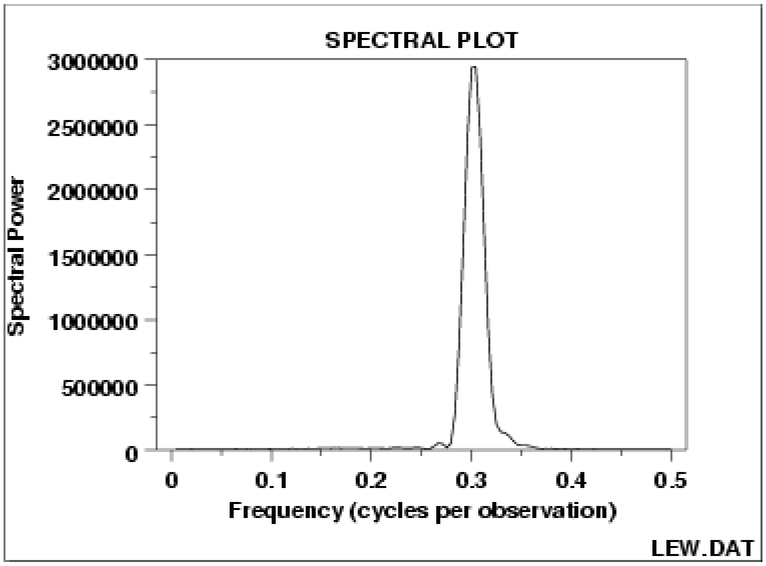
Then the system should be reexamined to find an explanation for the strong autocorrelation. Is it due to the

1. phenomenon under study; or
2. drifting in the environment; or
3. contamination from the data acquisition system (DAS)?

Oftentimes the source of the problem is item (3) above where contamination and carry-over from the data acquisition system result because the DAS does not have time to electronically recover before collecting the next data point. If this is the case, then consider slowing down the sampling rate to re-achieve randomness.

1.3.3.27.3. Spectral Plot: Sinusoidal Model

Spectral Plot for Sinusoidal Model



Conclusions

We can make the following conclusions from the above plot.

1. There is a single dominant peak at approximately 0.3.
2. There is an underlying single-cycle sinusoidal model.

Discussion

This spectral plot shows a single dominant frequency. This indicates that a single-cycle sinusoidal model might be appropriate.

If one were to naively assume that the data represented by the graph could be fit by the model

$$Y_i = A_0 + E_i$$

and then estimate the constant by the sample mean, the analysis would be incorrect because

- the sample mean is biased;
- the confidence interval for the mean, which is valid only for random data, is meaningless and too small.

On the other hand, the choice of the proper model

$$Y_i = C + \alpha \sin(2\pi\omega t_i + \phi) + E_i$$

where α is the amplitude, ω is the frequency (between 0 and .5 cycles per observation), and ϕ is the phase can be fit by non-linear least squares. The beam deflection data case study demonstrates fitting this type of model.

Recommended Next Steps

The recommended next steps are to:

1. Estimate the frequency from the spectral plot. This will be helpful as a starting value for the subsequent non-linear fitting. A complex demodulation phase plot can be used to fine tune the estimate of the frequency before performing the non-linear fit.

2. Do a complex demodulation amplitude plot to obtain an initial estimate of the amplitude and to determine if a constant amplitude is justified.

3. Carry out a non-linear fit of the model

$$Y_i = C + \alpha \sin(2\pi\omega t_i + \phi) + E_i$$

1.3.3.28. Standard Deviation Plot

Purpose: Detect Changes in Scale Between Groups

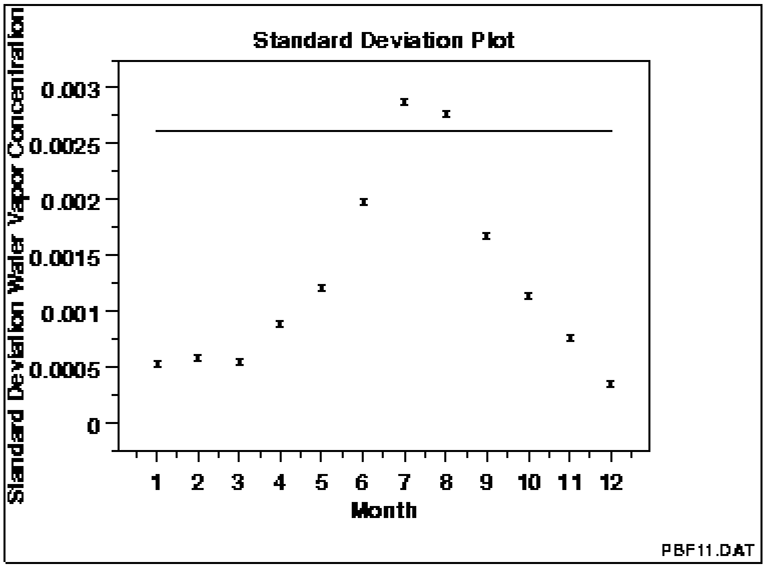
Standard deviation plots are used to see if the standard deviation varies between different groups of the data. The grouping is determined by the analyst. In most cases, the data provide a specific grouping variable. For example, the groups may be the levels of a factor variable. In the sample plot below, the months of the year provide the grouping.

Standard deviation plots can be used with ungrouped data to determine if the standard deviation is changing over time. In this case, the data are broken into an arbitrary number of equal-sized groups. For example, a data series with 400 points can be divided into 10 groups of 40 points each. A standard deviation plot can then be generated with these groups to see if the standard deviation is increasing or decreasing over time.

Although the standard deviation is the most commonly used measure of scale, the same concept applies to other measures of scale. For example, instead of plotting the standard deviation of each group, the median absolute deviation or the average absolute deviation might be plotted instead. This might be done if there were significant outliers in the data and a more robust measure of scale than the standard deviation was desired.

Standard deviation plots are typically used in conjunction with mean plots. The mean plot would be used to check for shifts in location while the standard deviation plot would be used to check for shifts in scale.

Sample Plot



This sample standard deviation plot shows

1. there is a shift in variation;
2. greatest variation is during the summer months.

Definition: Group Standard Deviations Versus Group ID

Standard deviation plots are formed by:

- Vertical axis: Group standard deviations
- Horizontal axis: Group identifier

A reference line is plotted at the overall standard deviation.

Questions

The standard deviation plot can be used to answer the following questions.

1. Are there any shifts in variation?
2. What is the magnitude of the shifts in variation?
3. Is there a distinct pattern in the shifts in variation?

Importance: Checking Assumptions

A common assumption in 1-factor analyses is that of equal variances. That is, the variance is the same for different levels of the factor variable. The standard deviation plot provides a graphical check for that assumption. A common assumption for univariate data is that the variance is constant. By grouping the data into equi-sized intervals, the standard deviation plot can provide a graphical test of this assumption.

Related Techniques

[Mean Plot](#)
[DOE Standard Deviation Plot](#)

Software

Most general purpose statistical software programs do not support a standard deviation plot. However, if the statistical program can generate the standard deviation for a group, it should be feasible to write a macro to generate this plot.

1.3.3.29. Star Plot

Purpose: Display Multivariate Data

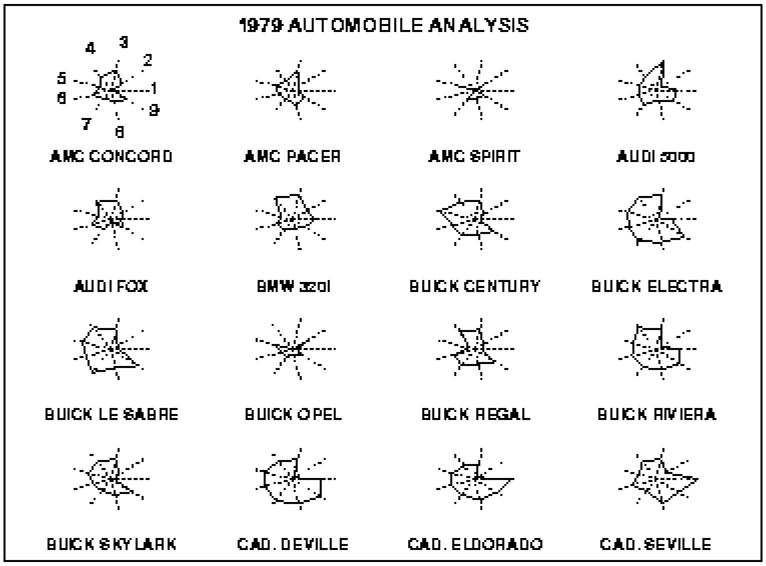
The star plot (Chambers 1983) is a method of displaying multivariate data. Each star represents a single observation. Typically, star plots are generated in a multi-plot format with many stars on each page and each star representing one observation.

Star plots are used to examine the relative values for a single data point (e.g., point 3 is large for variables 2 and 4, small for variables 1, 3, 5, and 6) and to locate similar points or dissimilar points.

Sample Plot

The plot below contains the star plots of 16 cars. The data file actually contains 74 cars, but we restrict the plot to what can reasonably be shown on one page. The variable list for the sample star plot is

1 Price
2 Mileage (MPG)
3 1978 Repair Record (1 = Worst, 5 = Best)
4 1977 Repair Record (1 = Worst, 5 = Best)
5 Headroom
6 Rear Seat Room
7 Trunk Space
8 Weight
9 Length



We can look at these plots individually or we can use them to identify clusters of cars with similar features. For example, we can look at the star plot of the Cadillac Seville and see that it is one of the most expensive cars, gets below average (but not among the worst) gas mileage, has an average repair record, and has average-to-above-average roominess and size. We can then compare the Cadillac models (the last three plots) with the AMC models (the first three plots). This comparison shows distinct patterns. The AMC models tend to be inexpensive, have below average gas mileage, and are small in both height and weight and in roominess. The Cadillac models are expensive, have poor gas mileage, and are large in both size and roominess.

Definition

The star plot consists of a sequence of equi-angular spokes, called radii, with each spoke representing one of the variables. The data length of a spoke is proportional to the magnitude of the variable for the data point relative to the maximum magnitude of the variable across all data points. A line is drawn connecting the data values for each spoke. This gives the plot a star-like appearance and the origin of the name of this plot.

Questions

The star plot can be used to answer the following questions:

1. What variables are dominant for a given observation?
2. Which observations are most similar, i.e., are there clusters of observations?
3. Are there outliers?

Weakness in Technique

Star plots are helpful for small-to-moderate-sized multivariate data sets. Their primary weakness is that their effectiveness is limited to data sets with less than a few hundred points. After that, they tend to be overwhelming.

Graphical techniques suited for large data sets are discussed by Scott.

Related Techniques

Alternative ways to plot multivariate data are discussed in Chambers, du Toit, and Everitt.

Software

Star plots are available in some general purpose statistical software progams.

1.3.3.30. Weibull Plot

Purpose: Graphical Check To See If Data Come From a Population That Would Be Fit by a Weibull Distribution

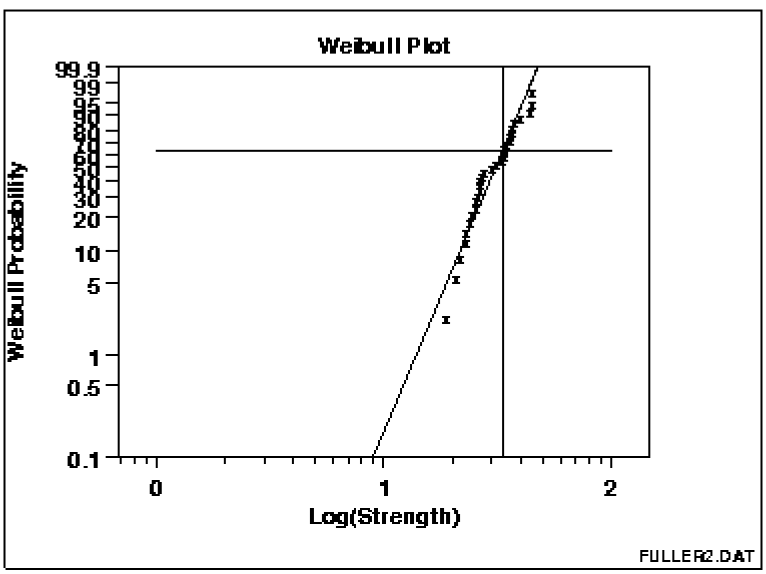
The Weibull plot (Nelson 1982) is a graphical technique for determining if a data set comes from a population that would logically be fit by a 2-parameter Weibull distribution (the location is assumed to be zero).

The Weibull plot has special scales that are designed so that if the data do in fact follow a Weibull distribution, the points will be linear (or nearly linear). The least squares fit of this line yields estimates for the shape and scale parameters of the Weibull distribution (the location is assumed to be zero).

Specifically, the shape parameter is the reciprocal of the slope of the fitted line and the scale parameter is the exponent of the intercept of the fitted line.

The Weibull distribution also has the property that the scale parameter falls at the 63.2% point irrespective of the value of the shape parameter. The plot shows a horizontal line at this 63.2% point and a vertical line where the horizontal line intersects the least squares fitted line. This vertical line shows the value of scale parameter.

Sample Plot



This Weibull plot shows that:

1. the assumption of a Weibull distribution is

reasonable;
2. the scale parameter estimate is computed to be 33.32;
3. the shape parameter estimate is computed to be 5.28; and
4. there are no outliers.

Note that the values on the x-axis ("0", "1", and "2") are the exponents. These actually denote the value $10^0 = 1$, $10^1 = 10$, and $10^2 = 100$.

Definition: Weibull Cumulative Probability Versus LN(Ordered Response)

The Weibull plot is formed by:

- Vertical axis: Weibull cumulative probability expressed as a percentage
- Horizontal axis: ordered failure times (in a LOG10 scale)

The vertical scale is ln(-ln(1-p)) where p=(i-0.3)/(n+0.4) and i is the rank of the observation. This scale is chosen in order to linearize the resulting plot for Weibull data.

Questions

The Weibull plot can be used to answer the following questions:

1. Do the data follow a 2-parameter Weibull distribution?
2. What is the best estimate of the shape parameter for the 2-parameter Weibull distribution?
3. What is the best estimate of the scale (= variation) parameter for the 2-parameter Weibull distribution?

Importance: Check Distributional Assumptions

Many statistical analyses, particularly in the field of reliability, are based on the assumption that the data follow a Weibull distribution. If the analysis assumes the data follow a Weibull distribution, it is important to verify this assumption and, if verified, find good estimates of the Weibull parameters.

Related Techniques

Weibull Probability Plot
Weibull PPCC Plot
Weibull Hazard Plot

The Weibull probability plot (in conjunction with the Weibull PPCC plot), the Weibull hazard plot, and the Weibull plot are all similar techniques that can be used for assessing the adequacy of the Weibull distribution as a model for the data, and additionally providing estimation for the shape, scale, or location parameters.

The Weibull hazard plot and Weibull plot are designed to handle censored data (which the Weibull probability plot does not).

Case Study	The Weibull plot is demonstrated in the fatigue life of aluminum alloy specimens case study.
Software	Weibull plots are generally available in statistical software programs that are designed to analyze reliability data.

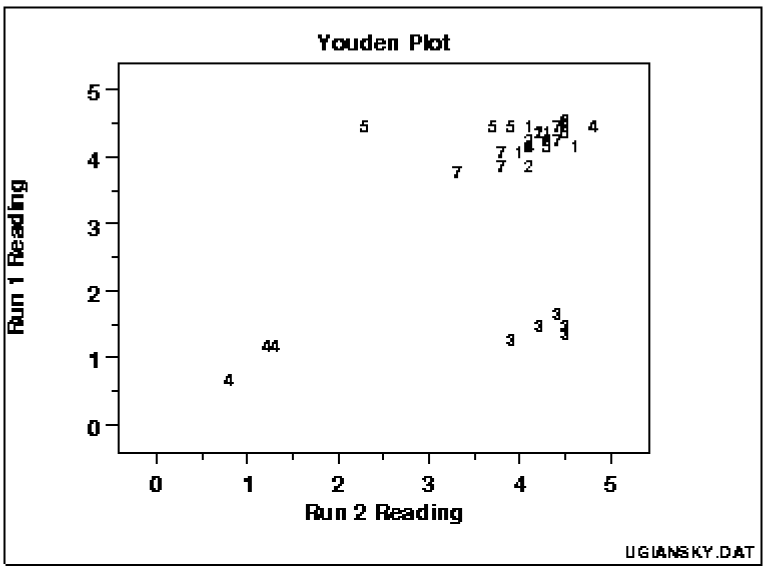
1.3.3.31. Youden Plot

Purpose: Interlab Comparisons

Youden plots are a graphical technique for analyzing interlab data when each lab has made two runs on the same product or one run on two different products.

The Youden plot is a simple but effective method for comparing both the within-laboratory variability and the between-laboratory variability.

Sample Plot



This plot shows:

1. Not all labs are equivalent.
2. Lab 4 is biased low.
3. Lab 3 has within-lab variability problems.
4. Lab 5 has an outlying run.

Definition: Response 1 Versus Response 2 Coded by Lab

Youden plots are formed by:

1. Vertical axis: Response variable 1 (i.e., run 1 or product 1 response value)
2. Horizontal axis: Response variable 2 (i.e., run 2 or product 2 response value)

In addition, the plot symbol is the lab id (typically an integer from 1 to *k* where *k* is the number of labs).

Sometimes a 45-degree reference line is drawn. Ideally, a lab generating two runs of the same product should produce reasonably similar results. Departures from this reference line indicate inconsistency from the lab. If two different products are being tested, then a 45-degree line may not be appropriate. However, if the labs are consistent, the points should lie near some fitted straight line.

Questions

The Youden plot can be used to answer the following questions:

1. Are all labs equivalent?
2. What labs have between-lab problems (reproducibility)?
3. What labs have within-lab problems (repeatability)?
4. What labs are outliers?

Importance

In interlaboratory studies or in comparing two runs from the same lab, it is useful to know if consistent results are generated. Youden plots should be a routine plot for analyzing this type of data.

DOE Youden Plot

The DOE Youden plot is a specialized Youden plot used in the design of experiments. In particular, it is useful for full and fractional designs.

Related Techniques

Scatter Plot

Software

The Youden plot is essentially a scatter plot, so it should be feasible to write a macro for a Youden plot in any general purpose statistical program that supports scatter plots.

1.3.3.31.1. DOE Youden Plot

DOE Youden Plot: Introduction

The DOE (Design of Experiments) Youden plot is a specialized Youden plot used in the analysis of full and fractional experiment designs. In particular, it is used in conjunction with the Yates algorithm. These designs may have a low level, coded as "-1" or "-", and a high level, coded as "+1" or "+", for each factor. In addition, there can optionally be one or more center points. Center points are at the midpoint between the low and high levels for each factor and are coded as "0".

The Yates agorithm and the the DOE Youden plot only use the "-1" and "+1" points. The Yates agorithm is used to estimate factor effects. The DOE Youden plot can be used to help determine the approriate model to based on the effect estimates from the Yates algorithm.

Construction of DOE Youden Plot

The following are the primary steps in the construction of the DOE Youden plot.

1. For a given factor or interaction term, compute the mean of the response variable for the low level of the factor and for the high level of the factor. Any center points are omitted from the computation.

2. Plot the point where the *y*-coordinate is the mean for the high level of the factor and the *x*-coordinate is the mean for the low level of the factor. The character used for the plot point should identify the factor or interaction term (e.g., "1" for factor 1, "13" for the interaction between factors 1 and 3).

3. Repeat steps 1 and 2 for each factor and interaction term of the data.

The high and low values of the interaction terms are obtained by multiplying the corresponding values of the main level factors. For example, the interaction term X_{13} is obtained by multiplying the values for X_1 with the corresponding values of X_3. Since the values for X_1 and X_3 are either "-1" or "+1", the resulting values for X_{13} are also either "-1" or "+1".

In summary, the DOE Youden plot is a plot of the mean of the response variable for the high level of a factor or interaction term against the mean of the response variable for the low level of that factor or interaction term.

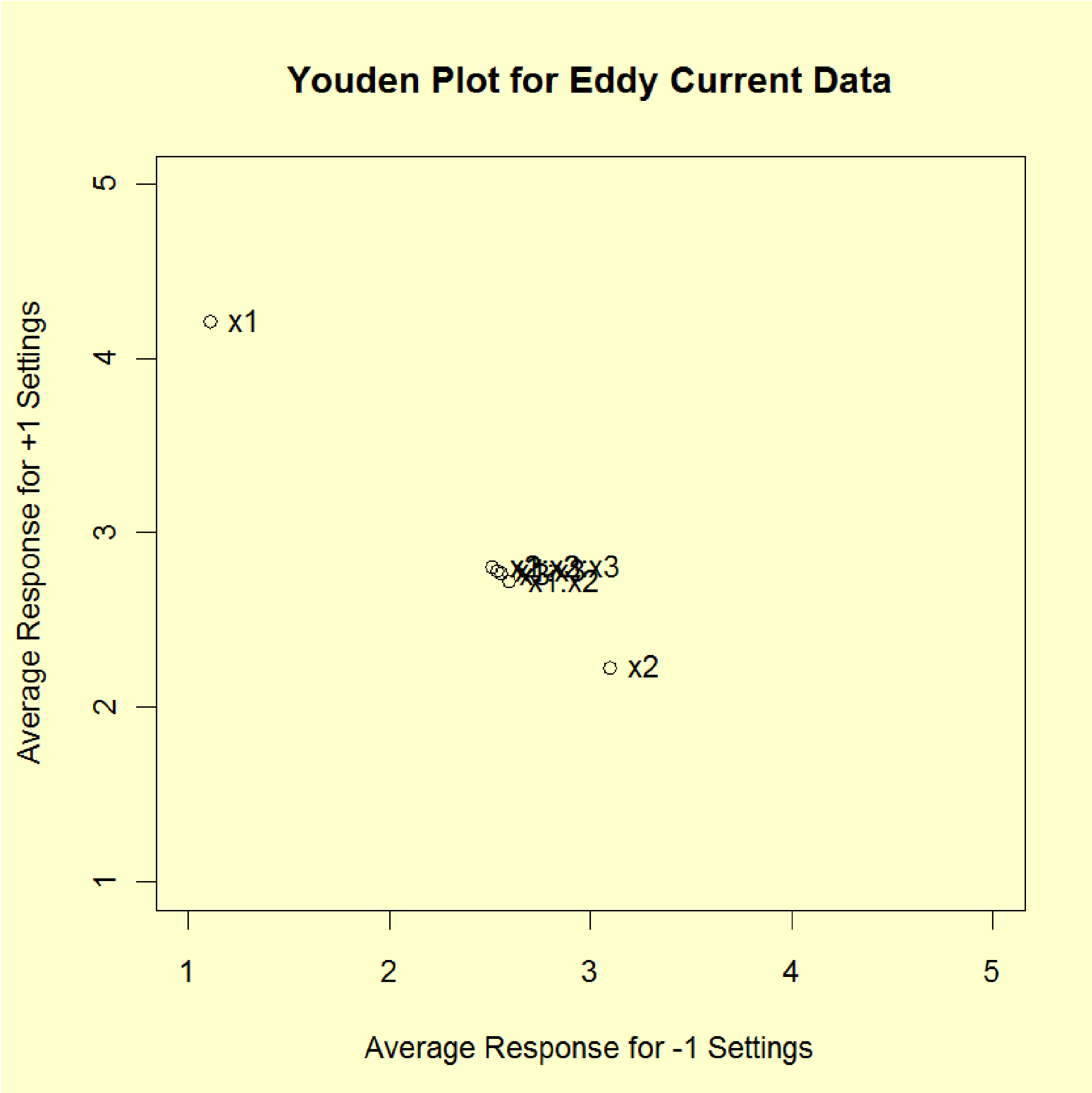
For unimportant factors and interaction terms, these mean values should be nearly the same. For important factors and interaction terms, these mean values should be quite different. So the interpretation of the plot is that unimportant factors should be clustered together near the grand mean. Points that stand apart from this cluster identify important factors that should be included in the model.

Sample DOE

The following is a DOE Youden plot for the data used in the Eddy current case study. The

Youden Plot

analysis in that case study demonstrated that X1 and X2 were the most important factors.



Interpretation of the Sample DOE Youden Plot

From the above DOE Youden plot, we see that factors 1 and 2 stand out from the others. That is, the mean response values for the low and high levels of factor 1 and factor 2 are quite different. For factor 3 and the 2 and 3-term interactions, the mean response values for the low and high levels are similar.

We would conclude from this plot that factors 1 and 2 are important and should be included in our final model while the remaining factors and interactions should be omitted from the final model.

Case Study

The Eddy current case study demonstrates the use of the DOE Youden plot in the context of the analysis of a full factorial design.

Software

DOE Youden plots are not typically available as built-in plots in statistical software programs. However, it should be relatively straightforward to write a macro to generate this plot in most general purpose statistical software programs.

1.3.3.32. 4-Plot

Purpose: Check Underlying Statistical Assumptions

The 4-plot is a collection of 4 specific EDA graphical techniques whose purpose is to test the assumptions that underlie most measurement processes. A 4-plot consists of a

1. run sequence plot;
2. lag plot;
3. histogram;
4. normal probability plot.

If the 4 underlying assumptions of a typical measurement process hold, then the above 4 plots will have a characteristic appearance (see the normal random numbers case study below); if any of the underlying assumptions fail to hold, then it will be revealed by an anomalous appearance in one or more of the plots. Several commonly encountered situations are demonstrated in the case studies below.

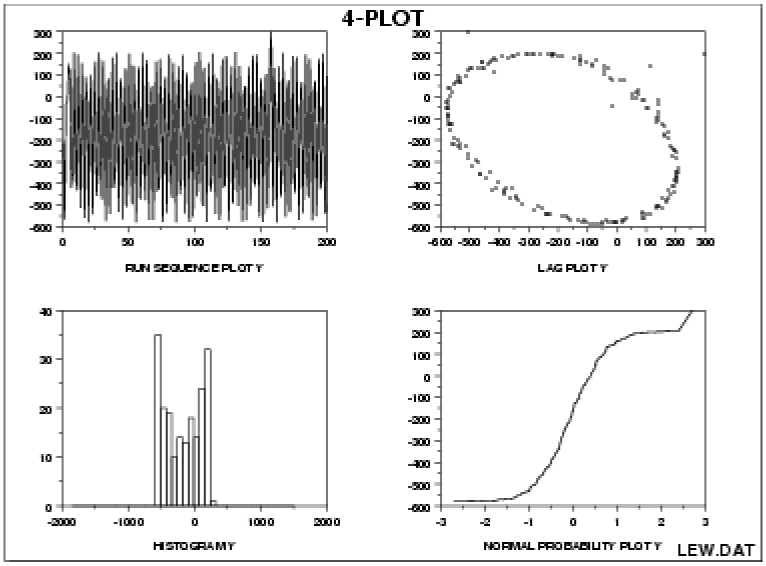
Although the 4-plot has an obvious use for univariate and time series data, its usefulness extends far beyond that. Many statistical models of the form

$$Y_i = f(X_1, ..., X_k) + E_i$$

have the same underlying assumptions for the error term. That is, no matter how complicated the functional fit, the assumptions on the underlying error term are still the same. The 4-plot can and should be routinely applied to the residuals when fitting models regardless of whether the model is simple or complicated.

Sample Plot: Process Has Fixed Location, Fixed Variation, Non-Random (Oscillatory), Non-Normal U-Shaped

Distribution, and Has 3 Outliers.



This 4-plot reveals the following:

1. the fixed location assumption is justified as shown by the run sequence plot in the upper left corner.
2. the fixed variation assumption is justified as shown by the run sequence plot in the upper left corner.
3. the randomness assumption is violated as shown by the non-random (oscillatory) lag plot in the upper right corner.
4. the assumption of a common, normal distribution is violated as shown by the histogram in the lower left corner and the normal probability plot in the lower right corner. The distribution is non-normal and is a U-shaped distribution.
5. there are several outliers apparent in the lag plot in the upper right corner.

Definition: 1. Run Sequence Plot; 2. Lag Plot; 3. Histogram; 4. Normal Probability Plot

The 4-plot consists of the following:

1. Run sequence plot to test fixed location and variation.
   - Vertically: Y_i
   - Horizontally: i
2. Lag Plot to test randomness.
   - Vertically: Y_i
   - Horizontally: Y_{i-1}
3. Histogram to test (normal) distribution.
   - Vertically: Counts
   - Horizontally: Y
4. Normal probability plot to test normal distribution.
   - Vertically: Ordered Y_i
   - Horizontally: Theoretical values from a normal N(0,1) distribution for ordered Y_i

Questions

4-plots can provide answers to many questions:

1. Is the process in-control, stable, and predictable?
2. Is the process drifting with respect to location?
3. Is the process drifting with respect to variation?
4. Are the data random?
5. Is an observation related to an adjacent observation?
6. If the data are a time series, is is white noise?
7. If the data are a time series and not white noise, is it sinusoidal, autoregressive, etc.?
8. If the data are non-random, what is a better model?
9. Does the process follow a normal distribution?
10. If non-normal, what distribution does the process follow?
11. Is the model

    $$Y_i = A_0 + E_i$$

    valid and sufficient?
12. If the default model is insufficient, what is a better model?
13. Is the formula $s_{\bar{Y}} = s/\sqrt{N}$ valid?
14. Is the sample mean a good estimator of the process location?
15. If not, what would be a better estimator?
16. Are there any outliers?

Importance: Testing Underlying Assumptions Helps Ensure the Validity of the Final Scientific and Engineering Conclusions

There are 4 assumptions that typically underlie all measurement processes; namely, that the data from the process at hand "behave like":

1. random drawings;
2. from a fixed distribution;
3. with that distribution having a fixed location; and
4. with that distribution having fixed variation.

Predictability is an all-important goal in science and engineering. If the above 4 assumptions hold, then we have achieved probabilistic predictability--the ability to make probability statements not only about the process in the past, but also about the process in the future. In short, such processes are said to be "statistically in control". If the 4 assumptions do not hold, then we have a process that is drifting (with respect to location, variation, or distribution), is unpredictable, and is out of control. A simple characterization of such processes by a location estimate, a variation estimate, or a distribution "estimate" inevitably leads to optimistic and grossly invalid engineering conclusions.

Inasmuch as the validity of the final scientific and engineering conclusions is inextricably linked to the validity of these same 4 underlying assumptions, it naturally follows that there is a real necessity for all 4 assumptions to be routinely tested. The 4-plot (run sequence plot, lag plot, histogram, and normal probability plot) is seen as a simple, efficient, and powerful way of carrying out this routine checking.

Interpretation: Flat, Equi-Banded, Random, Bell-Shaped, and Linear

Of the 4 underlying assumptions:

1. If the fixed location assumption holds, then the run sequence plot will be flat and non-drifting.
2. If the fixed variation assumption holds, then the vertical spread in the run sequence plot will be approximately the same over the entire horizontal axis.
3. If the randomness assumption holds, then the lag plot will be structureless and random.
4. If the fixed distribution assumption holds (in particular, if the fixed normal distribution assumption holds), then the histogram will be bell-shaped and the normal probability plot will be approximatelylinear.

If all 4 of the assumptions hold, then the process is "statistically in control". In practice, many processes fall short of achieving this ideal.

Related Techniques

Run Sequence Plot
Lag Plot
Histogram
Normal Probability Plot

Autocorrelation Plot
Spectral Plot
PPCC Plot

Case Studies

The 4-plot is used in most of the case studies in this chapter:

1. Normal random numbers (the ideal)
2. Uniform random numbers
3. Random walk
4. Josephson junction cryothermometry
5. Beam deflections
6. Filter transmittance
7. Standard resistor
8. Heat flow meter 1

Software

It should be feasible to write a macro for the 4-plot in any general purpose statistical software program that supports the capability for multiple plots per page and supports the underlying plot techniques.

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.3. Graphical Techniques: Alphabetic

1.3.3.33. 6-Plot

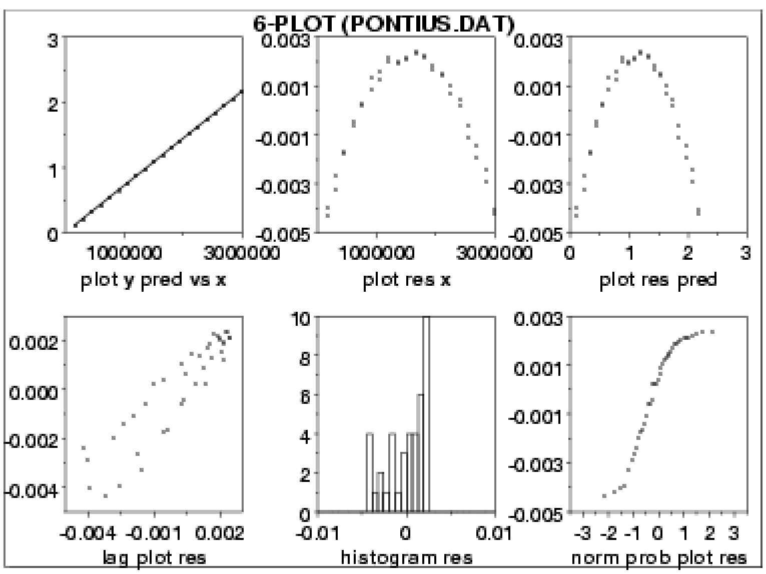
Purpose: Graphical Model Validation

The 6-plot is a collection of 6 specific graphical techniques whose purpose is to assess the validity of a Y versus X fit. The fit can be a linear fit, a non-linear fit, a LOWESS (locally weighted least squares) fit, a spline fit, or any other fit utilizing a single independent variable.

The 6 plots are:

1. Scatter plot of the response and predicted values versus the independent variable;
2. Scatter plot of the residuals versus the independent variable;
3. Scatter plot of the residuals versus the predicted values;
4. Lag plot of the residuals;
5. Histogram of the residuals;
6. Normal probability plot of the residuals.

Sample Plot



This 6-plot, which followed a linear fit, shows that the linear model is not adequate. It suggests that a quadratic model would be a better model.

Definition: 6

The 6-plot consists of the following:

Component Plots

1. Response and predicted values
   - Vertical axis: Response variable, predicted values
   - Horizontal axis: Independent variable
2. Residuals versus independent variable
   - Vertical axis: Residuals
   - Horizontal axis: Independent variable
3. Residuals versus predicted values
   - Vertical axis: Residuals
   - Horizontal axis: Predicted values
4. Lag plot of residuals
   - Vertical axis: RES(I)
   - Horizontal axis: RES(I-1)
5. Histogram of residuals
   - Vertical axis: Counts
   - Horizontal axis: Residual values
6. Normal probability plot of residuals
   - Vertical axis: Ordered residuals
   - Horizontal axis: Theoretical values from a normal N(0,1) distribution for ordered residuals

Questions

The 6-plot can be used to answer the following questions:

1. Are the residuals approximately normally distributed with a fixed location and scale?
2. Are there outliers?
3. Is the fit adequate?
4. Do the residuals suggest a better fit?

Importance: Validating Model

A model involving a response variable and a single independent variable has the form:

$$Y_i = f(X_i) + E_i$$

where Y is the response variable, X is the independent variable, *f* is the linear or non-linear fit function, and E is the random component. For a good model, the error component should behave like:

1. random drawings (i.e., independent);
2. from a fixed distribution;
3. with fixed location; and
4. with fixed variation.

In addition, for fitting models it is usually further assumed that the fixed distribution is normal and the fixed location is zero. For a good model the fixed variation should be as small as possible. A necessary component of fitting models is to verify these assumptions for the error component and to assess whether the variation for the error component is sufficiently small. The histogram, lag plot, and normal probability plot are used to verify the fixed distribution, location, and variation assumptions on the error component. The plot of the response variable and the predicted values versus the independent variable is used to assess whether the variation is sufficiently small. The plots of the residuals versus the independent variable and the predicted values is used to assess the independence assumption.

Assessing the validity and quality of the fit in terms of the above assumptions is an absolutely vital part of the model-fitting process. No fit should be considered complete without an adequate model validation step.

Related Techniques

Linear Least Squares
Non-Linear Least Squares
Scatter Plot
Run Sequence Plot
Lag Plot
Normal Probability Plot
Histogram

Case Study

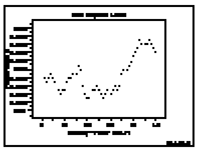
The 6-plot is used in the Alaska pipeline data case study.

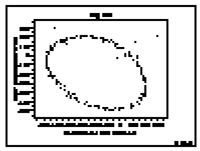
Software

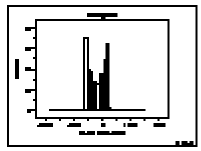
It should be feasible to write a macro for the 6-plot in any general purpose statistical software program that supports the capability for multiple plots per page and supports the underlying plot techniques.

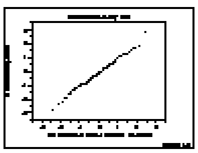
1.3.4. Graphical Techniques: By Problem Category

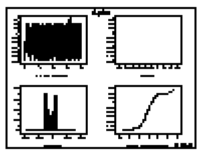
Univariate
y = c + e

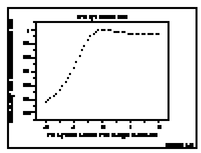

Run Sequence Plot: 1.3.3.25

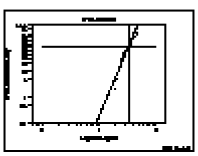

Lag Plot: 1.3.3.15

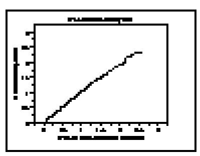

Histogram: 1.3.3.14

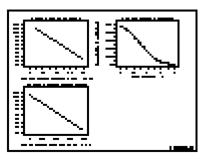

Normal Probability Plot: 1.3.3.21

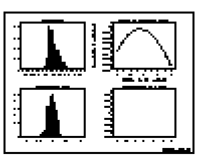

4-Plot: 1.3.3.32

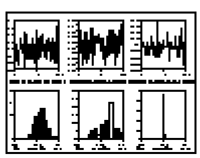

PPCC Plot: 1.3.3.23


Weibull Plot: 1.3.3.30

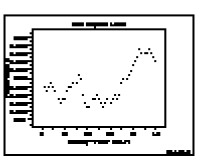

Probability Plot: 1.3.3.22

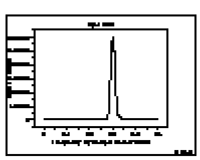

Box-Cox Linearity Plot: 1.3.3.5

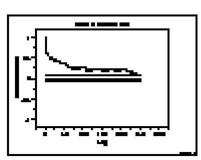

Box-Cox Normality Plot: 1.3.3.6

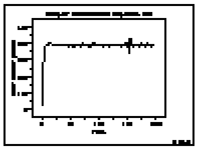

Bootstrap Plot: 1.3.3.4

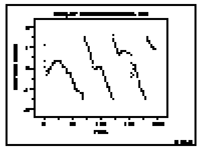
Time Series
y = f(t) + e


Run Sequence Plot: 1.3.3.25


Spectral Plot: 1.3.3.27

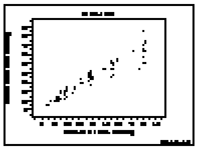

Autocorrelation Plot: 1.3.3.1

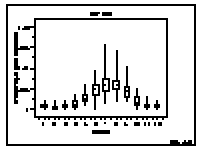

Complex Demodulation Amplitude Plot: 1.3.3.8

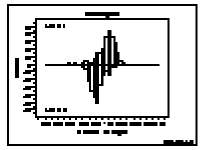

Complex Demodulation Phase Plot: 1.3.3.9

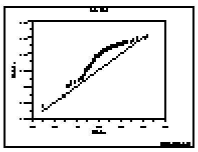
---

1 Factor
y = f(x) + e

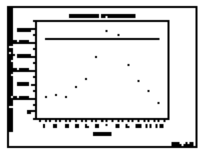

Scatter Plot: 1.3.3.26


Box Plot: 1.3.3.7


Bihistogram: 1.3.3.2

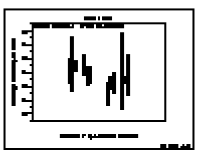

Quantile-Quantile Plot: 1.3.3.24


Mean Plot: 1.3.3.20

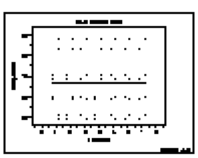

Standard Deviation Plot: 1.3.3.28

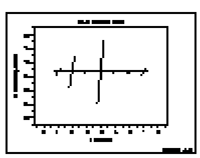
Multi-Factor/Comparative

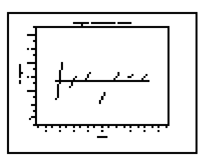
y = f(xp, x1,x2,...,xk) + e

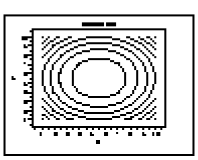

Block Plot: 1.3.3.3

Multi-Factor/Screening
y = f(x1,x2,x3,...,xk) + e

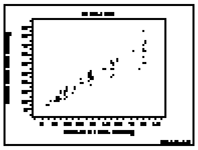

DOE Scatter Plot: 1.3.3.11

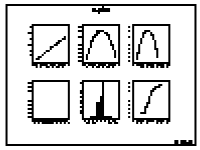

DOE Mean Plot: 1.3.3.12

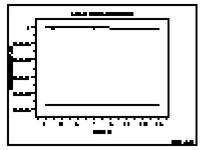

DOE Standard Deviation Plot: 1.3.3.13

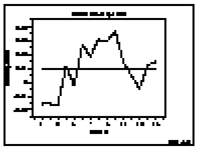

Contour Plot: 1.3.3.10

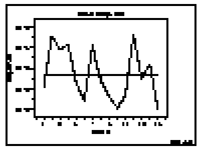
Regression
y = f(x1,x2,x3,...,xk) + e

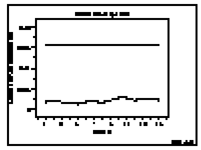

Scatter Plot: 1.3.3.26


6-Plot: 1.3.3.33

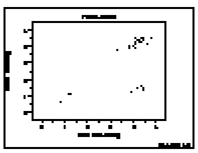

Linear Correlation Plot: 1.3.3.16


Linear Intercept Plot: 1.3.3.17

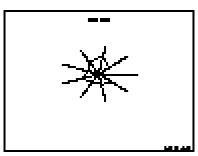

Linear Slope Plot: 1.3.3.18


Linear Residual Standard Deviation Plot:1.3.3.19

Interlab
(y1,y2) = f(x) + e


Youden Plot: 1.3.3.31

Multivariate
(y1,y2,...,yp)


Star Plot: 1.3.3.29

1.3.5. Quantitative Techniques

Confirmatory Statistics

The techniques discussed in this section are classical statistical methods as opposed to EDA techniques. EDA and classical techniques are not mutually exclusive and can be used in a complementary fashion. For example, the analysis can start with some simple graphical techniques such as the 4-plot followed by the classical confirmatory methods discussed herein to provide more rigorous statements about the conclusions. If the classical methods yield different conclusions than the graphical analysis, then some effort should be invested to explain why. Often this is an indication that some of the assumptions of the classical techniques are violated.

Many of the quantitative techniques fall into two broad categories:

1. Interval estimation
2. Hypothesis tests

Interval Estimates

It is common in statistics to estimate a parameter from a sample of data. The value of the parameter using all of the possible data, not just the sample data, is called the population parameter or true value of the parameter. An estimate of the true parameter value is made using the sample data. This is called a point estimate or a sample estimate.

For example, the most commonly used measure of location is the mean. The population, or true, mean is the sum of all the members of the given population divided by the number of members in the population. As it is typically impractical to measure every member of the population, a random sample is drawn from the population. The sample mean is calculated by summing the values in the sample and dividing by the number of values in the sample. This sample mean is then used as the point estimate of the population mean.

Interval estimates expand on point estimates by incorporating the uncertainty of the point estimate. In the example for the mean above, different samples from the same population will generate different values for the sample mean. An interval estimate quantifies this uncertainty in the sample estimate by computing lower and upper values of an interval which will, with a given level of confidence (i.e., probability), contain the population parameter.

Hypothesis Tests

Hypothesis tests also address the uncertainty of the sample estimate. However, instead of providing an interval, a hypothesis test attempts to refute a specific claim about a population parameter based on the sample data. For example, the hypothesis might be one of the following:

- the population mean is equal to 10
- the population standard deviation is equal to 5
- the means from two populations are equal
- the standard deviations from 5 populations are equal

To reject a hypothesis is to conclude that it is false. However, to accept a hypothesis does not mean that it is true, only that we do not have evidence to believe otherwise. Thus hypothesis tests are usually stated in terms of both a condition that is doubted (null hypothesis) and a condition that is believed (alternative hypothesis).

A common format for a hypothesis test is:

H_0:	A statement of the null hypothesis, e.g., two population means are equal.
H_a:	A statement of the alternative hypothesis, e.g., two population means are not equal.
Test Statistic:	The test statistic is based on the specific hypothesis test.
Significance Level:	The significance level, α, defines the sensitivity of the test. A value of α = 0.05 means that we inadvertently reject the null hypothesis 5% of the time when it is in fact true. This is also called the type I error. The choice of α is somewhat arbitrary, although in practice values of 0.1, 0.05, and 0.01 are commonly used. The probability of rejecting the null hypothesis when it is in fact false is called the power of the test and is denoted by 1 - β. Its complement, the probability of accepting the null hypothesis when the alternative hypothesis is, in fact, true (type II error), is called β and can only be computed for a specific alternative hypothesis.
Critical Region:	The critical region encompasses those values of the test statistic that lead to a rejection of the null hypothesis. Based on the distribution of the test statistic and the significance level,

a cut-off value for the test statistic is computed. Values either above or below or both (depending on the direction of the test) this cut-off define the critical region.

Practical Versus Statistical Significance

It is important to distinguish between statistical significance and practical significance. Statistical significance simply means that we reject the null hypothesis. The ability of the test to detect differences that lead to rejection of the null hypothesis depends on the sample size. For example, for a particularly large sample, the test may reject the null hypothesis that two process means are equivalent. However, in practice the difference between the two means may be relatively small to the point of having no real engineering significance. Similarly, if the sample size is small, a difference that is large in engineering terms may not lead to rejection of the null hypothesis. The analyst should not just blindly apply the tests, but should combine engineering judgement with statistical analysis.

Bootstrap Uncertainty Estimates

In some cases, it is possible to mathematically derive appropriate uncertainty intervals. This is particularly true for intervals based on the assumption of a normal distribution. However, there are many cases in which it is not possible to mathematically derive the uncertainty. In these cases, the bootstrap provides a method for empirically determining an appropriate interval.

Table of Contents

Some of the more common classical quantitative techniques are listed below. This list of quantitative techniques is by no means meant to be exhaustive. Additional discussions of classical statistical techniques are contained in the product comparisons chapter.

1.3.5.1. Measures of Location

Location

A fundamental task in many statistical analyses is to estimate a location parameter for the distribution; i.e., to find a typical or central value that best describes the data.

Definition of Location

The first step is to define what we mean by a typical value. For univariate data, there are three common definitions:

1. mean - the mean is the sum of the data points divided by the number of data points. That is,

$$\bar{Y} = \sum_{i=1}^{N} Y_i / N$$

   The mean is that value that is most commonly referred to as the average. We will use the term average as a synonym for the mean and the term typical value to refer generically to measures of location.

2. median - the median is the value of the point which has half the data smaller than that point and half the data larger than that point. That is, if $X_1, X_2, \ldots, X_N$ is a random sample sorted from smallest value to largest value, then the median is defined as:

$$\tilde{Y} = Y_{(N+1)/2} \qquad \text{if } N \text{ is odd}$$

$$\tilde{Y} = (Y_{N/2} + Y_{(N/2)+1})/2 \qquad \text{if } N \text{ is even}$$

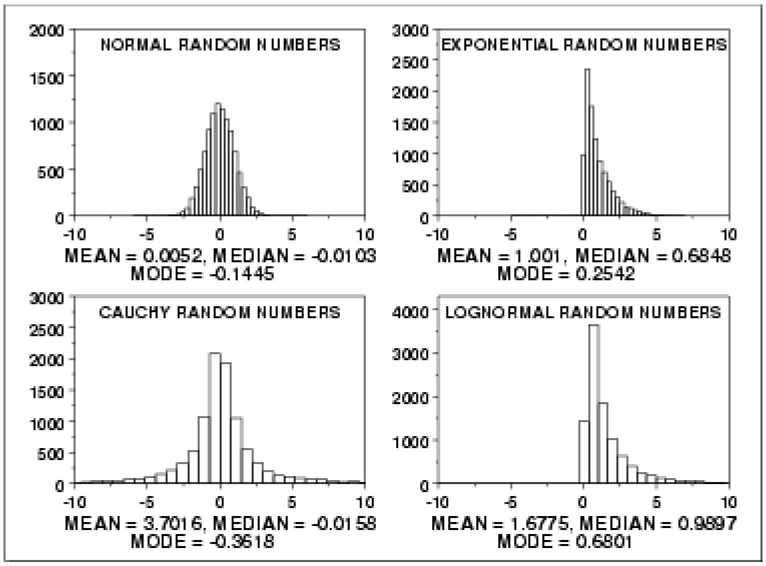
3. mode - the mode is the value of the random sample that occurs with the greatest frequency. It is not necessarily unique. The mode is typically used in a qualitative fashion. For example, there may be a single dominant hump in the data perhaps two or more smaller humps in the data. This is usually evident from a histogram of the data.

   When taking samples from continuous populations, we need to be somewhat careful in how we define the mode. That is, any specific value may not occur more than once if the data are continuous. What may be a more meaningful, if less exact measure, is the midpoint of the class interval of the histogram with the highest peak.

Why Different Measures

A natural question is why we have more than one measure of the typical value. The following example helps to explain why these alternative definitions are useful and necessary.

This plot shows histograms for 10,000 random numbers generated from a normal, an exponential, a Cauchy, and a lognormal distribution.



Normal Distribution

The first histogram is a sample from a normal distribution. The mean is 0.005, the median is -0.010, and the mode is -0.144 (the mode is computed as the midpoint of the histogram interval with the highest peak).

The normal distribution is a symmetric distribution with well-behaved tails and a single peak at the center of the distribution. By symmetric, we mean that the distribution can be folded about an axis so that the 2 sides coincide. That is, it behaves the same to the left and right of some center point. For a normal distribution, the mean, median, and mode are actually equivalent. The histogram above generates similar estimates for the mean, median, and mode. Therefore, if a histogram or normal probability plot indicates that your data are approximated well by a normal distribution, then it is reasonable to use the mean as the location estimator.

Exponential Distribution

The second histogram is a sample from an exponential distribution. The mean is 1.001, the median is 0.684, and the mode is 0.254 (the mode is computed as the midpoint of the histogram interval with the highest peak).

The exponential distribution is a skewed, i. e., not symmetric, distribution. For skewed distributions, the mean and median are not the same. The mean will be pulled in the direction of the skewness. That is, if the right tail is heavier than the left tail, the mean will be greater than the median. Likewise, if the left tail is heavier than the right tail, the mean will be less than the median.

For skewed distributions, it is not at all obvious whether the mean, the median, or the mode is the more meaningful measure of the typical value. In this case, all three measures are useful.

Cauchy Distribution

The third histogram is a sample from a Cauchy distribution. The mean is 3.70, the median is -0.016, and the mode is -0.362 (the mode is computed as the midpoint of the histogram interval with the highest peak).

For better visual comparison with the other data sets, we restricted the histogram of the Cauchy distribution to values between -10 and 10. The full Cauchy data set in fact has a minimum of approximately -29,000 and a maximum of approximately 89,000.

The Cauchy distribution is a symmetric distribution with heavy tails and a single peak at the center of the distribution. The Cauchy distribution has the interesting property that collecting more data does not provide a more accurate estimate of the mean. That is, the sampling distribution of the mean is equivalent to the sampling distribution of the original data. This means that for the Cauchy distribution the mean is useless as a measure of the typical value. For this histogram, the mean of 3.7 is well above the vast majority of the data. This is caused by a few very extreme values in the tail. However, the median does provide a useful measure for the typical value.

Although the Cauchy distribution is an extreme case, it does illustrate the importance of heavy tails in measuring the mean. Extreme values in the tails distort the mean. However, these extreme values do not distort the median since the median is based on ranks. In general, for data with extreme values in the tails, the median provides a better estimate of location than does the mean.

Lognormal Distribution

The fourth histogram is a sample from a lognormal distribution. The mean is 1.677, the median is 0.989, and the mode is 0.680 (the mode is computed as the midpoint of the histogram interval with the highest peak).

The lognormal is also a skewed distribution. Therefore the mean and median do not provide similar estimates for the location. As with the exponential distribution, there is no obvious answer to the question of which is the more meaningful measure of location.

Robustness

There are various alternatives to the mean and median for measuring location. These alternatives were developed to address non-normal data since the mean is an optimal estimator if in fact your data are normal.

Tukey and Mosteller defined two types of robustness where robustness is a lack of susceptibility to the effects of nonnormality.

1. Robustness of validity means that the confidence intervals for the population location have a 95% chance of covering the population location regardless of what the underlying distribution is.
2. Robustness of efficiency refers to high effectiveness in the face of non-normal tails. That is, confidence intervals for the population location tend to be almost as narrow as the best that could be done if we knew the true shape of the distributuion.

The mean is an example of an estimator that is the best we can do if the underlying distribution is normal. However, it lacks robustness of validity. That is, confidence intervals based on the mean tend not to be precise if the underlying distribution is in fact not normal.

The median is an example of a an estimator that tends to have robustness of validity but not robustness of efficiency.

The alternative measures of location try to balance these two concepts of robustness. That is, the confidence intervals for the case when the data are normal should be almost as narrow as the confidence intervals based on the mean. However, they should maintain their validity even if the underlying data are not normal. In particular, these alternatives address the problem of heavy-tailed distributions.

Alternative Measures of Location

A few of the more common alternative location measures are:

1. Mid-Mean - computes a mean using the data between the 25th and 75th percentiles.
2. Trimmed Mean - similar to the mid-mean except different percentile values are used. A common choice is to trim 5% of the points in both the lower and upper tails, i.e., calculate the mean for data between the 5th and 95th percentiles.
3. Winsorized Mean - similar to the trimmed mean. However, instead of trimming the points, they are set to the lowest (or highest) value. For example, all data below the 5th percentile are set equal to the value of the 5th percentile and all data greater than the 95th percentile are set equal to the 95th percentile.
4. Mid-range = (smallest + largest)/2.

The first three alternative location estimators defined above have the advantage of the median in the sense that they are not unduly affected by extremes in the tails. However, they generate estimates that are closer to the mean for data that are normal (or nearly so).

The mid-range, since it is based on the two most extreme points, is not robust. Its use is typically restricted to situations in which the behavior at the extreme points is relevant.

Case Study

The uniform random numbers case study compares the performance of several different location estimators for a particular non-normal distribution.

Software

Most general purpose statistical software programs can compute at least some of the measures of location discussed above.

1.3.5.2. Confidence Limits for the Mean

Purpose: Interval Estimate for Mean

Confidence limits for the mean (Snedecor and Cochran, 1989) are an interval estimate for the mean. Interval estimates are often desirable because the estimate of the mean varies from sample to sample. Instead of a single estimate for the mean, a confidence interval generates a lower and upper limit for the mean. The interval estimate gives an indication of how much uncertainty there is in our estimate of the true mean. The narrower the interval, the more precise is our estimate.

Confidence limits are expressed in terms of a confidence coefficient. Although the choice of confidence coefficient is somewhat arbitrary, in practice 90 %, 95 %, and 99 % intervals are often used, with 95 % being the most commonly used.

As a technical note, a 95 % confidence interval does **not** mean that there is a 95 % probability that the interval contains the true mean. The interval computed from a given sample either contains the true mean or it does not. Instead, the level of confidence is associated with the method of calculating the interval. The confidence coefficient is simply the proportion of samples of a given size that may be expected to contain the true mean. That is, for a 95 % confidence interval, if many samples are collected and the confidence interval computed, in the long run about 95 % of these intervals would contain the true mean.

Definition: Confidence Interval

Confidence limits are defined as:

$$\bar{Y} \pm t_{1-\alpha/2,\,N-1} \frac{s}{\sqrt{N}}$$

where $\bar{Y}$ is the sample mean, s is the sample standard deviation, N is the sample size, α is the desired significance level, and $t_{1-\alpha/2,\,N-1}$ is the $100(1-\alpha/2)$ percentile of the *t* distribution with $N - 1$ degrees of freedom. Note that the confidence coefficient is $1 - \alpha$.

From the formula, it is clear that the width of the interval is controlled by two factors:

1. As N increases, the interval gets narrower from the $\sqrt{N}$ term.

   That is, one way to obtain more precise estimates for the mean is to increase the sample size.

2. The larger the sample standard deviation, the larger the confidence interval. This simply means that noisy data, i.e., data with a large standard deviation, are going to generate wider intervals than data with a smaller standard deviation.

Definition: Hypothesis Test

To test whether the population mean has a specific value, μ_0, against the two-sided alternative that it does not have a value μ_0, the confidence interval is converted to hypothesis-test form. The test is a one-sample *t*-test, and it is defined as:

H_0:	$\mu = \mu_0$
H_a:	$\mu \neq \mu_0$
Test Statistic:	$T = (\bar{Y} - \mu_0)/(s/\sqrt{N})$ where $\bar{Y}$, *N*, and s are defined as above.
Significance Level:	α. The most commonly used value for α is 0.05.
Critical Region:	Reject the null hypothesis that the mean is a specified value, μ_0, if $T < t_{\alpha/2,\, N-1}$ or $T > t_{1-\alpha/2,\, N-1}$

Confidence Interval Example

We generated a 95 %, two-sided confidence interval for the ZARR13.DAT data set based on the following information.

```
N                                = 195
MEAN                             =   9.261460
STANDARD DEVIATION               =   0.022789
t1-0.025,N-1                       =   1.9723

LOWER LIMIT = 9.261460 - 1.9723*0.022789/√195
UPPER LIMIT = 9.261460 + 1.9723*0.022789/√195
```

Thus, a 95 % confidence interval for the mean is (9.258242, 9.264679).

t-Test Example

We performed a two-sided, one-sample *t*-test using the ZARR13.DAT data set to test the null hypothesis that the population mean is equal to 5.

```
H0:  μ = 5
Ha:  μ ≠ 5

Test statistic:  T = 2611.284
Degrees of freedom:  ν = 194
Significance level:  α = 0.05
Critical value:  t1-α/2,ν = 1.9723
Critical region:  Reject H0 if |T| > 1.9723
```

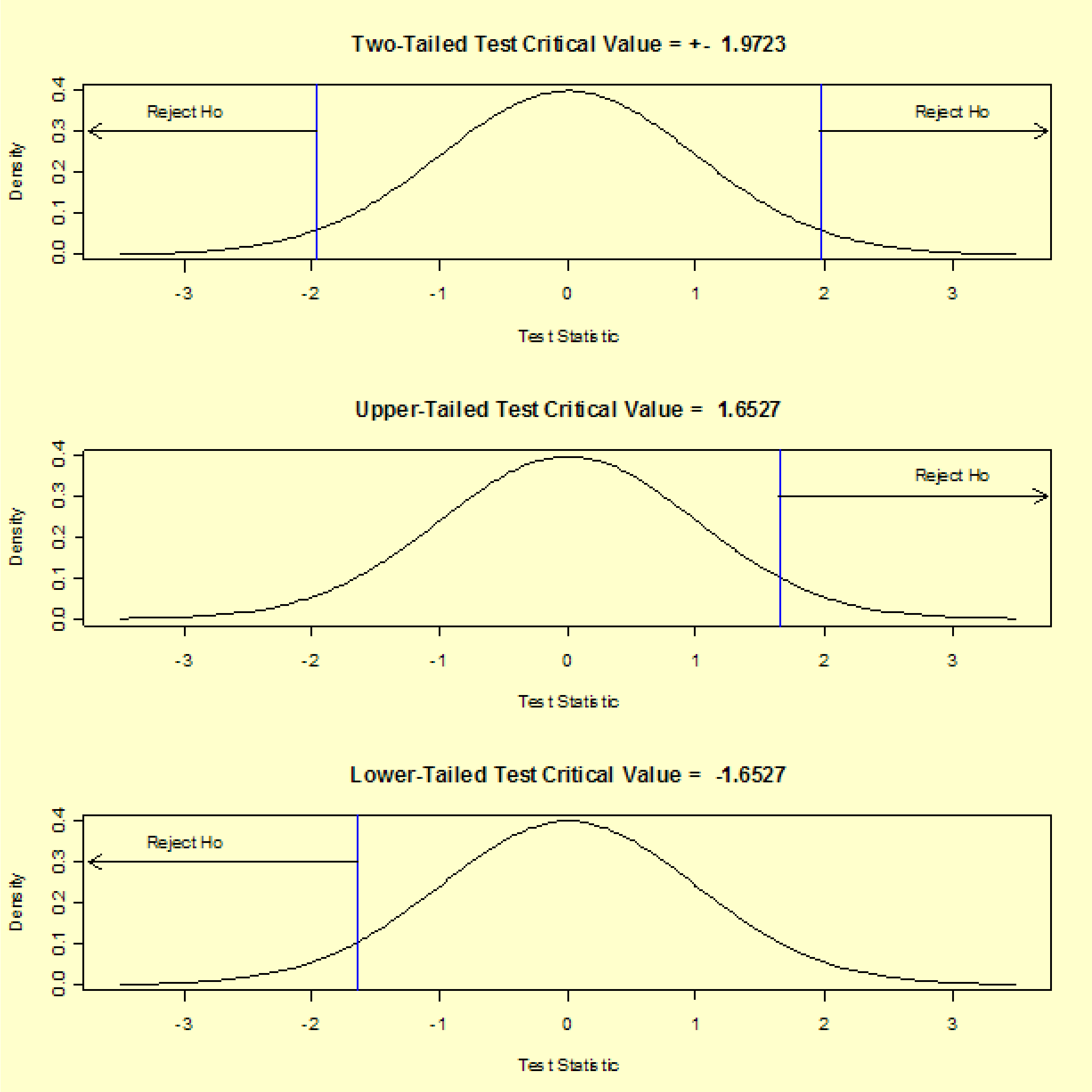
We reject the null hypotheses for our two-tailed *t*-test because the absolute value of the test statistic is greater than the critical value. If we were to perform an upper, one-tailed test, the critical value would be $t_{1-\alpha,\nu}$ = 1.6527, and we would still reject the null hypothesis.

The confidence interval provides an alternative to the hypothesis test. If the confidence interval contains 5, then H_0 cannot be rejected. In our example, the confidence interval (9.258242, 9.264679) does not contain 5, indicating that the population mean does not equal 5 at the 0.05 level of significance.

In general, there are three possible alternative hypotheses and rejection regions for the one-sample *t*-test:

Alternative Hypothesis	Rejection Region
H_a: $\mu \neq \mu_0$	$\|T\| > t_{1-\alpha/2,\nu}$
H_a: $\mu > \mu_0$	$T > t_{1-\alpha,\nu}$
H_a: $\mu < \mu_0$	$T < t_{\alpha,\nu}$

The rejection regions for three posssible alternative hypotheses using our example data are shown in the following graphs.



Questions

Confidence limits for the mean can be used to answer the following questions:

1. What is a reasonable estimate for the mean?
2. How much variability is there in the estimate of the mean?
3. Does a given target value fall within the confidence limits?

Related Techniques

Two-Sample *t*-Test

Confidence intervals for other location estimators such as the median or mid-mean tend to be mathematically difficult or intractable. For these cases, confidence intervals can be obtained using the bootstrap.

Case Study

Heat flow meter data.

Software

Confidence limits for the mean and one-sample *t*-tests are available in just about all general purpose statistical software programs. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.3. Two-Sample *t*-Test for Equal Means

Purpose: Test if two population means are equal

The two-sample *t*-test (Snedecor and Cochran, 1989) is used to determine if two population means are equal. A common application is to test if a new process or treatment is superior to a current process or treatment.

There are several variations on this test.

1. The data may either be paired or not paired. By paired, we mean that there is a one-to-one correspondence between the values in the two samples. That is, if $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are the two samples, then X_i corresponds to Y_i. For paired samples, the difference $X_i - Y_i$ is usually calculated. For unpaired samples, the sample sizes for the two samples may or may not be equal. The formulas for paired data are somewhat simpler than the formulas for unpaired data.
2. The variances of the two samples may be assumed to be equal or unequal. Equal variances yields somewhat simpler formulas, although with computers this is no longer a significant issue.
3. In some applications, you may want to adopt a new process or treatment only if it exceeds the current treatment by some threshold. In this case, we can state the null hypothesis in the form that the difference between the two populations means is equal to some constant ($\mu_1 - \mu_2 = d_0$) where the constant is the desired threshold.

Definition

The two-sample *t*-test for unpaired data is defined as:

H_0:	$\mu_1 = \mu_2$
H_a:	$\mu_1 \neq \mu_2$
Test Statistic:	$T = \frac{\bar{Y}_1 - \bar{Y}_2}{\sqrt{s_1^2/N_1 + s_2^2/N_2}}$

where N_1 and N_2 are the sample sizes, $\bar{Y}_1$ and $\bar{Y}_2$ are the sample means, and s_1^2 and s_2^2 are the sample variances.

If equal variances are assumed, then the formula reduces to:

$$T = \frac{\bar{Y}_1 - \bar{Y}_2}{s_p\sqrt{1/N_1 + 1/N_2}}$$

where

$$s_p^2 = \frac{(N_1 - 1)s_1^2 + (N_2 - 1)s_2^2}{N_1 + N_2 - 2}$$

Significance Level: α.

Critical Region: Reject the null hypothesis that the two means are equal if

$$|T| > t_{1-\alpha/2,\nu}$$

where $t_{1-\alpha/2,\nu}$ is the critical value of the *t* distribution with ν degrees of freedom where

$$\nu = \frac{(s_1^2/N_1 + s_2^2/N_2)^2}{(s_1^2/N_1)^2/(N_1 - 1) + (s_2^2/N_2)^2/(N_2 - 1)}$$

If equal variances are assumed, then

$$\nu = N_1 + N_2 - 2$$

Two-Sample t-Test Example

The following two-sample *t*-test was generated for the AUTO83B.DAT data set. The data set contains miles per gallon for U.S. cars (sample 1) and for Japanese cars (sample 2); the summary statistics for each sample are shown below.

```
SAMPLE 1:
    NUMBER OF OBSERVATIONS      = 249
    MEAN                         =  20.14458
    STANDARD DEVIATION          =    6.41470
    STANDARD ERROR OF THE MEAN  =    0.40652

SAMPLE 2:
    NUMBER OF OBSERVATIONS      = 79
    MEAN                         = 30.48101
    STANDARD DEVIATION          =  6.10771
    STANDARD ERROR OF THE MEAN  =  0.68717
```

We are testing the hypothesis that the population means are equal for the two samples. We assume that the variances for the two samples are equal.

H_0: $\mu_1 = \mu_2$
H_a: $\mu_1 \ne \mu_2$

Test statistic: T = -12.62059
Pooled standard deviation: s_p = 6.34260
Degrees of freedom: ν = 326
Significance level: α = 0.05
Critical value (upper tail): $t_{1-\alpha/2,\nu}$ = 1.9673
Critical region: Reject H_0 if $|T|$ > 1.9673

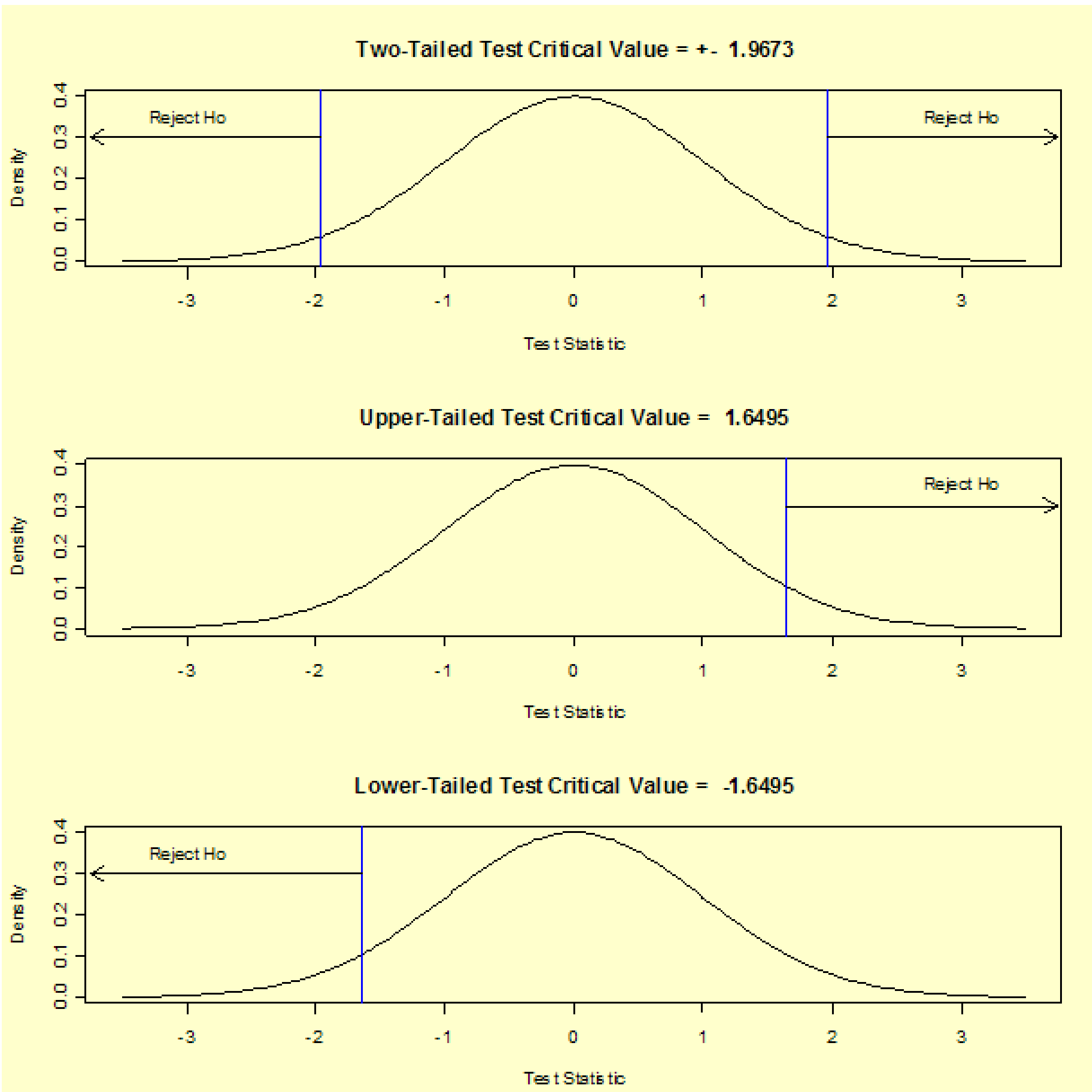
The absolute value of the test statistic for our example, 12.62059, is greater than the critical value of 1.9673, so we reject the null hypothesis and conclude that the two population means are different at the 0.05 significance level.

In general, there are three possible alternative hypotheses and rejection regions for the one-sample *t*-test:

Alternative Hypothesis	Rejection Region

H_a: $\mu_1 \neq \mu_2$	$\|T\| > t_{1-\alpha/2,\nu}$
H_a: $\mu_1 > \mu_2$	$T > t_{1-\alpha,\nu}$
H_a: $\mu_1 < \mu_2$	$T < t_{\alpha,\nu}$

For our two-tailed *t*-test, the critical value is $t_{1-\alpha/2,\nu}$ = 1.9673, where α = 0.05 and ν = 326. If we were to perform an upper, one-tailed test, the critical value would be $t_{1-\alpha,\nu}$ = 1.6495. The rejection regions for three posssible alternative hypotheses using our example data are shown below.



Questions

Two-sample *t*-tests can be used to answer the following questions:

1. Is process 1 equivalent to process 2?
2. Is the new process better than the current process?
3. Is the new process better than the current process by at least some pre-determined threshold amount?

Related Techniques	Confidence Limits for the Mean Analysis of Variance
Case Study	Ceramic strength data.
Software	Two-sample *t*-tests are available in just about all general purpose statistical software programs. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.3.1. Data Used for Two-Sample *t*-Test

Data Used for Two-Sample t-Test Example

The following is the data used for the two-sample *t*-test example. The first column is miles per gallon for U.S. cars and the second column is miles per gallon for Japanese cars. For the *t*-test example, rows with the second column equal to -999 were deleted.

```
   18             24
   15             27
   18             27
   16             25
   17             31
   15             35
   14             24
   14             19
   14             28
   15             23
   15             27
   14             20
   15             22
   14             18
   22             20
   18             31
   21             32
   21             31
   10             32
   10             24
   11             26
    9             29
   28             24
   25             24
   19             33
   16             33
   17             32
   19             28
   18             19
   14             32
   14             34
   14             26
   14             30
   12             22
   13             22
   13             33
   18             39
   22             36
   19             28
   18             27
   23             21
   26             24
   25             30
   20             34
   21             32
   13             38
   14             37
   15             30
   14             31
   17             37
   11             32
   13             47
   12             41
```

```
     13          45
     15          34
     13          33
     13          24
     14          32
     22          39
     28          35
     13          32
     14          37
     13          38
     14          34
     15          34
     12          32
     13          33
     13          32
     14          25
     13          24
     12          37
     13          31
     18          36
     16          36
     18          34
     18          38
     23          32
     11          38
     12          32
     13        -999
     12        -999
     18        -999
     21        -999
     19        -999
     21        -999
     15        -999
     16        -999
     15        -999
     11        -999
     20        -999
     21        -999
     19        -999
     15        -999
     26        -999
     25        -999
     16        -999
     16        -999
     18        -999
     16        -999
     13        -999
     14        -999
     14        -999
     14        -999
     28        -999
     19        -999
     18        -999
     15        -999
     15        -999
     16        -999
     15        -999
     16        -999
     14        -999
     17        -999
     16        -999
     15        -999
     18        -999
     21        -999
     20        -999
     13        -999
     23        -999
     20        -999
     23        -999
     18        -999
     19        -999
     25        -999
     26        -999
     18        -999
     16        -999
     16        -999
     15        -999
     22        -999
     22        -999
     24        -999
     23        -999
```

```
           29          -999
           25          -999
           20          -999
           18          -999
           19          -999
           18          -999
           27          -999
           13          -999
           17          -999
           13          -999
           13          -999
           13          -999
           30          -999
           26          -999
           18          -999
           17          -999
           16          -999
           15          -999
           18          -999
           21          -999
           19          -999
           19          -999
           16          -999
           16          -999
           16          -999
           16          -999
           25          -999
           26          -999
           31          -999
           34          -999
           36          -999
           20          -999
           19          -999
           20          -999
           19          -999
           21          -999
           20          -999
           25          -999
           21          -999
           19          -999
           21          -999
           21          -999
           19          -999
           18          -999
           19          -999
           18          -999
           18          -999
           18          -999
           30          -999
           31          -999
           23          -999
           24          -999
           22          -999
           20          -999
           22          -999
           20          -999
           21          -999
           17          -999
           18          -999
           17          -999
           18          -999
           17          -999
           16          -999
           19          -999
           19          -999
           36          -999
           27          -999
           23          -999
           24          -999
           34          -999
           35          -999
           28          -999
           29          -999
           27          -999
           34          -999
           32          -999
           28          -999
           26          -999
           24          -999
           19          -999
           28          -999
```

24	-999
27	-999
27	-999
26	-999
24	-999
30	-999
39	-999
35	-999
34	-999
30	-999
22	-999
27	-999
20	-999
18	-999
28	-999
27	-999
34	-999
31	-999
29	-999
27	-999
24	-999
23	-999
38	-999
36	-999
25	-999
38	-999
26	-999
22	-999
36	-999
27	-999
27	-999
32	-999
28	-999
31	-999

1.3.5.4. One-Factor ANOVA

Purpose: Test for Equal Means Across Groups

One factor analysis of variance (Snedecor and Cochran, 1989) is a special case of analysis of variance (ANOVA), for one factor of interest, and a generalization of the two-sample *t*-test. The two-sample *t*-test is used to decide whether two groups (levels) of a factor have the same mean. One-way analysis of variance generalizes this to levels where *k*, the number of levels, is greater than or equal to 2.

For example, data collected on, say, five instruments have one factor (instruments) at five levels. The ANOVA tests whether instruments have a significant effect on the results.

Definition

The Product and Process Comparisons chapter (chapter 7) contains a more extensive discussion of one-factor ANOVA, including the details for the mathematical computations of one-way analysis of variance.

The model for the analysis of variance can be stated in two mathematically equivalent ways. In the following discussion, each level of each factor is called a cell. For the one-way case, a cell and a level are equivalent since there is only one factor. In the following, the subscript i refers to the level and the subscript j refers to the observation within a level. For example, Y_{23} refers to the third observation in the second level.

The first model is

$$Y_{ij} = \mu_i + E_{ij}$$

This model decomposes the response into a mean for each cell and an error term. The analysis of variance provides estimates for each cell mean. These estimated cell means are the predicted values of the model and the differences between the response variable and the estimated cell means are the residuals. That is

$$\hat{Y}_{ij} = \hat{\mu}_i$$

$$R_{ij} = Y_{ij} - \hat{\mu}_i$$

The second model is

$$Y_{ij} = \mu + \alpha_i + E_{ij}$$

This model decomposes the response into an overall (grand) mean, the effect of the *i*th factor level, and an error term. The analysis of variance provides estimates of the grand mean and the effect of the *i*th factor level. The predicted values and the residuals of the model are

$$\hat{Y}_{ij} = \hat{\mu} + \hat{\alpha}_i$$

$$R_{ij} = Y_{ij} - \hat{\mu} - \hat{\alpha}_i$$

The distinction between these models is that the second model divides the cell mean into an overall mean and the effect of the *i*th factor level. This second model makes the factor effect more explicit, so we will emphasize this approach.

Model Validation

Note that the ANOVA model assumes that the error term, E_{ij}, should follow the assumptions for a univariate measurement process. That is, after performing an analysis of variance, the model should be validated by analyzing the residuals.

One-Way ANOVA Example

A one-way analysis of variance was generated for the GEAR.DAT data set. The data set contains 10 measurements of gear diameter for ten different batches for a total of 100 measurements.

SOURCE	DEGREES OF FREEDOM	SUM OF SQUARES	MEAN SQUARE	*F* STATISTIC
BATCH	9	0.000729	0.000081	2.2969
RESIDUAL	90	0.003174	0.000035	
TOTAL (CORRECTED)	99	0.003903	0.000039	

RESIDUAL STANDARD DEVIATION = 0.00594

BATCH	N	MEAN	SD(MEAN)
1	10	0.99800	0.00178
2	10	0.99910	0.00178
3	10	0.99540	0.00178
4	10	0.99820	0.00178
5	10	0.99190	0.00178
6	10	0.99880	0.00178
7	10	1.00150	0.00178
8	10	1.00040	0.00178
9	10	0.99830	0.00178
10	10	0.99480	0.00178

The ANOVA table decomposes the variance into the following component sum of squares:

- Total sum of squares. The degrees of freedom for this

entry is the number of observations minus one.
- Sum of squares for the factor. The degrees of freedom for this entry is the number of levels minus one. The mean square is the sum of squares divided by the number of degrees of freedom.
- Residual sum of squares. The degrees of freedom is the total degrees of freedom minus the factor degrees of freedom. The mean square is the sum of squares divided by the number of degrees of freedom.

The sums of squares summarize how much of the variance in the data (total sum of squares) is accounted for by the factor effect (batch sum of squares) and how much is random error (residual sum of squares). Ideally, we would like most of the variance to be explained by the factor effect.

The ANOVA table provides a formal *F* test for the factor effect. For our example, we are testing the following hypothesis.

H_0: All individual batch means are equal.
H_a: At least one batch mean is not equal to the others.

The *F* statistic is the batch mean square divided by the residual mean square. This statistic follows an *F* distribution with (k-1) and (N-k) degrees of freedom. For our example, the critical *F* value (upper tail) for α = 0.05, (k-1) = 10, and (N-k) = 90 is 1.9376. Since the *F* statistic, 2.2969, is greater than the critical value, we conclude that there is a significant batch effect at the 0.05 level of significance.

Once we have determined that there is a significant batch effect, we might be interested in comparing individual batch means. The batch means and the standard errors of the batch means provide some information about the individual batches, however, we may want to employ multiple comparison methods for a more formal analysis. (See Box, Hunter, and Hunter for more information.)

In addition to the quantitative ANOVA output, it is recommended that any analysis of variance be complemented with model validation. At a minimum, this should include:

1. a run sequence plot of the residuals,
2. a normal probability plot of the residuals, and
3. a scatter plot of the predicted values against the residuals.

Question

The analysis of variance can be used to answer the following question

- Are means the same across groups in the data?

Importance	The analysis of uncertainty depends on whether the factor significantly affects the outcome.
Related Techniques	Two-sample *t*-test Multi-factor analysis of variance Regression Box plot
Software	Most general purpose statistical software programs can generate an analysis of variance. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.5. Multi-factor Analysis of Variance

Purpose: Detect significant factors

The analysis of variance (ANOVA) (Neter, Wasserman, and Kunter, 1990) is used to detect significant factors in a multi-factor model. In the multi-factor model, there is a response (dependent) variable and one or more factor (independent) variables. This is a common model in designed experiments where the experimenter sets the values for each of the factor variables and then measures the response variable.

Each factor can take on a certain number of values. These are referred to as the levels of a factor. The number of levels can vary betweeen factors. For designed experiments, the number of levels for a given factor tends to be small. Each factor and level combination is a cell. Balanced designs are those in which the cells have an equal number of observations and unbalanced designs are those in which the number of observations varies among cells. It is customary to use balanced designs in designed experiments.

Definition

The Product and Process Comparisons chapter (chapter 7) contains a more extensive discussion of two-factor ANOVA, including the details for the mathematical computations.

The model for the analysis of variance can be stated in two mathematically equivalent ways. We explain the model for a two-way ANOVA (the concepts are the same for additional factors). In the following discussion, each combination of factors and levels is called a cell. In the following, the subscript i refers to the level of factor 1, j refers to the level of factor 2, and the subscript k refers to the kth observation within the (i,j)th cell. For example, Y_{235} refers to the fifth observation in the second level of factor 1 and the third level of factor 2.

The first model is

$$Y_{ijk} = \mu_{ij} + E_{ijk}$$

This model decomposes the response into a mean for each cell and an error term. The analysis of variance provides estimates for each cell mean. These cell means are the predicted values of the model and the differences between the response variable and the estimated cell means are the residuals. That is

$$\hat{Y}_{ijk} = \hat{\mu}_{ij}$$

$$R_{ijk} = Y_{ijk} - \hat{\mu}_{ij}$$

The second model is

$$Y_{ijk} = \mu + \alpha_i + \beta_j + E_{ijk}$$

This model decomposes the response into an overall (grand) mean, factor effects ($\hat{\alpha}_i$ and $\hat{\beta}_j$ represent the effects of the *i*th level of the first factor and the *j*th level of the second factor, respectively), and an error term. The analysis of variance provides estimates of the grand mean and the factor effects. The predicted values and the residuals of the model are

$$\hat{Y}_{ijk} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j$$

$$R_{ijk} = Y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j$$

The distinction between these models is that the second model divides the cell mean into an overall mean and factor effects. This second model makes the factor effect more explicit, so we will emphasize this approach.

Model Validation

Note that the ANOVA model assumes that the error term, E_{ijk}, should follow the assumptions for a univariate measurement process. That is, after performing an analysis of variance, the model should be validated by analyzing the residuals.

Multi-Factor ANOVA Example

An analysis of variance was performed for the JAHANMI2.DAT data set. The data contains four, two-level factors: table speed, down feed rate, wheel grit size, and batch. There are 30 measurements of ceramic strength for each factor combination for a total of 480 measurements.

```
 SOURCE              DF SUM OF SQUARES    MEAN
SQUARE   F STATISTIC
 -------------------------------------------------
----------------
 TABLE SPEED           1   26672.726562
26672.726562        6.7080
 DOWN FEED RATE        1   11524.053711
11524.053711        2.8982
 WHEEL GRIT SIZE       1   14380.633789
14380.633789        3.6166
 BATCH                 1  727143.125000
727143.125000      182.8703
 RESIDUAL            475 1888731.500000
3976.276855
 TOTAL (CORRECTED)   479 2668446.000000
5570.868652

 RESIDUAL STANDARD DEVIATION = 63.05772781

 FACTOR           LEVEL    N       MEAN      SD(MEAN)
 -----------------------------------------------------
 TABLE SPEED        -1    240   657.53168     2.87818
                     1    240   642.62286     2.87818
 DOWN FEED RATE     -1    240   645.17755     2.87818
                     1    240   654.97723     2.87818
```

```
WHEEL GRIT SIZE  -1    240  655.55084    2.87818
                  1    240  644.60376    2.87818
BATCH             1    240  688.99890    2.87818
                  2    240  611.15594    2.87818
```

The ANOVA decomposes the variance into the following component sum of squares:

- Total sum of squares. The degrees of freedom for this entry is the number of observations minus one.
- Sum of squares for each of the factors. The degrees of freedom for these entries are the number of levels for the factor minus one. The mean square is the sum of squares divided by the number of degrees of freedom.
- Residual sum of squares. The degrees of freedom is the total degrees of freedom minus the sum of the factor degrees of freedom. The mean square is the sum of squares divided by the number of degrees of freedom.

The analysis of variance summarizes how much of the variance in the data (total sum of squares) is accounted for by the factor effects (factor sum of squares) and how much is due to random error (residual sum of squares). Ideally, we would like most of the variance to be explained by the factor effects. The ANOVA table provides a formal *F* test for the factor effects. To test the overall batch effect in our example we use the following hypotheses.

H_0: All individual batch means are equal.
H_a: At least one batch mean is not equal to the others.

The *F* statistic is the mean square for the factor divided by the residual mean square. This statistic follows an *F* distribution with $(k-1)$ and $(N-k)$ degrees of freedom where k is the number of levels for the given factor. Here, we see that the size of the "direction" effect dominates the size of the other effects. For our example, the critical *F* value (upper tail) for $\alpha = 0.05$, $(k-1) = 1$, and $(N-k) = 475$ is 3.86111. Thus, "table speed" and "batch" are significant at the 5 % level while "down feed rate" and "wheel grit size" are not significant at the 5 % level.

In addition to the quantitative ANOVA output, it is recommended that any analysis of variance be complemented with model validation. At a minimum, this should include

1. A run sequence plot of the residuals.
2. A normal probability plot of the residuals.
3. A scatter plot of the predicted values against the residuals.

Questions

The analysis of variance can be used to answer the following questions:

1. Do any of the factors have a significant effect?

2. Which is the most important factor?
3. Can we account for most of the variability in the data?

Related Techniques

One-factor analysis of variance
Two-sample *t*-test
Box plot
Block plot
DOE mean plot

Case Study

The quantitative ANOVA approach can be contrasted with the more graphical EDA approach in the ceramic strength case study.

Software

Most general purpose statistical software programs can perform multi-factor analysis of variance. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.6. Measures of Scale

Scale, Variability, or Spread

A fundamental task in many statistical analyses is to characterize the *spread*, or variability, of a data set. Measures of scale are simply attempts to estimate this variability.

When assessing the variability of a data set, there are two key components:

1. How spread out are the data values near the center?
2. How spread out are the tails?

Different numerical summaries will give different weight to these two elements. The choice of scale estimator is often driven by which of these components you want to emphasize.

The histogram is an effective graphical technique for showing both of these components of the spread.

Definitions of Variability

For univariate data, there are several common numerical measures of the spread:

1. variance - the variance is defined as

$$s^2 = \sum_{i=1}^{N}(Y_i - \bar{Y})^2/(N-1)$$

where $\bar{Y}$ is the mean of the data.

The variance is roughly the arithmetic average of the squared distance from the mean. Squaring the distance from the mean has the effect of giving greater weight to values that are further from the mean. For example, a point 2 units from the mean adds 4 to the above sum while a point 10 units from the mean adds 100 to the sum. Although the variance is intended to be an overall measure of spread, it can be greatly affected by the tail behavior.

2. standard deviation - the standard deviation is the square root of the variance. That is,

$$s = \sqrt{\sum_{i=1}^{N}(Y_i - \bar{Y})^2/(N-1)}$$

The standard deviation restores the units of the spread to the original data units (the variance squares the units).

3. range - the range is the largest value minus the smallest value in a data set. Note that this measure is based only on the lowest and highest extreme values in the sample. The spread near the center of the data is not captured at all.

4. average absolute deviation - the average absolute deviation (AAD) is defined as

$$AAD = \sum_{i=1}^{N}(|Y_i - \bar{Y}|)/N$$

where $\bar{Y}$ is the mean of the data and $|Y|$ is the absolute value of Y. This measure does not square the distance from the mean, so it is less affected by extreme observations than are the variance and standard deviation.

5. median absolute deviation - the median absolute deviation (MAD) is defined as

$$MAD = median(|Y_i - \tilde{Y}|)$$

where $\tilde{Y}$ is the median of the data and $|Y|$ is the absolute value of Y. This is a variation of the average absolute deviation that is even less affected by extremes in the tail because the data in the tails have less influence on the calculation of the median than they do on the mean.

6. interquartile range - this is the value of the 75th percentile minus the value of the 25th percentile. This measure of scale attempts to measure the variability of points near the center.

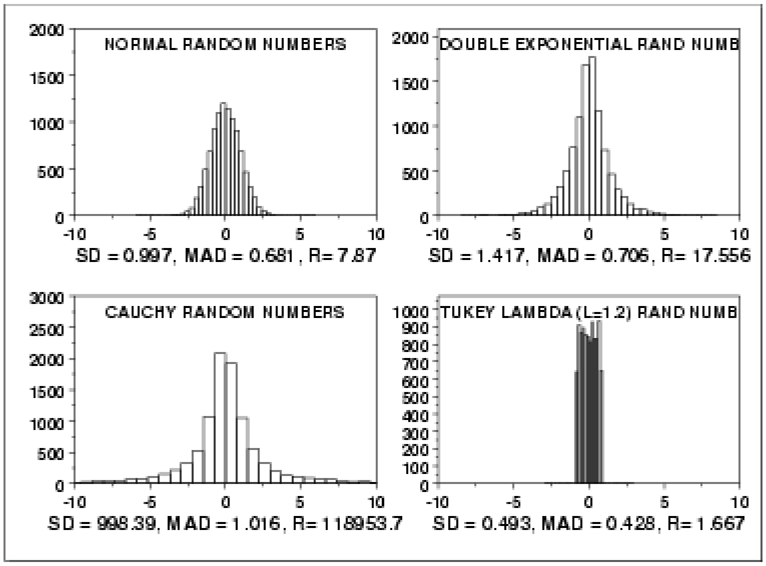
In summary, the variance, standard deviation, average absolute deviation, and median absolute deviation measure both aspects of the variability; that is, the variability near the center and the variability in the tails. They differ in that the average absolute deviation and median absolute deviation do not give undue weight to the tail behavior. On the other hand, the range only uses the two most extreme points and the interquartile range only uses the middle portion of the data.

Why Different

The following example helps to clarify why these alternative defintions of spread are useful and necessary.

Measures?

This plot shows histograms for 10,000 random numbers generated from a normal, a double exponential, a Cauchy, and a Tukey-Lambda distribution.



Normal Distribution

The first histogram is a sample from a normal distribution. The standard deviation is 0.997, the median absolute deviation is 0.681, and the range is 7.87.

The normal distribution is a symmetric distribution with well-behaved tails and a single peak at the center of the distribution. By symmetric, we mean that the distribution can be folded about an axis so that the two sides coincide. That is, it behaves the same to the left and right of some center point. In this case, the median absolute deviation is a bit less than the standard deviation due to the downweighting of the tails. The range of a little less than 8 indicates the extreme values fall within about 4 standard deviations of the mean. If a histogram or normal probability plot indicates that your data are approximated well by a normal distribution, then it is reasonable to use the standard deviation as the spread estimator.

Double Exponential Distribution

The second histogram is a sample from a double exponential distribution. The standard deviation is 1.417, the median absolute deviation is 0.706, and the range is 17.556.

Comparing the double exponential and the normal histograms shows that the double exponential has a stronger peak at the center, decays more rapidly near the center, and has much longer tails. Due to the longer tails, the standard deviation tends to be inflated compared to the normal. On the other hand, the median absolute deviation is only slightly larger than it is for the normal data. The longer tails are clearly reflected in the value of the range, which shows that the extremes fall about 6 standard deviations from the mean compared to about 4 for the normal data.

Cauchy Distribution

The third histogram is a sample from a Cauchy distribution. The standard deviation is 998.389, the median absolute deviation is 1.16, and the range is 118,953.6.

The Cauchy distribution is a symmetric distribution with heavy tails and a single peak at the center of the distribution. The Cauchy distribution has the interesting property that collecting more data does not provide a more accurate estimate for the mean or standard deviation. That is, the sampling distribution of the means and standard deviation are equivalent to the sampling distribution of the original data. That means that for the Cauchy distribution the standard deviation is useless as a measure of the spread. From the histogram, it is clear that just about all the data are between about -5 and 5. However, a few very extreme values cause both the standard deviation and range to be extremely large. However, the median absolute deviation is only slightly larger than it is for the normal distribution. In this case, the median absolute deviation is clearly the better measure of spread.

Although the Cauchy distribution is an extreme case, it does illustrate the importance of heavy tails in measuring the spread. Extreme values in the tails can distort the standard deviation. However, these extreme values do not distort the median absolute deviation since the median absolute deviation is based on ranks. In general, for data with extreme values in the tails, the median absolute deviation or interquartile range can provide a more stable estimate of spread than the standard deviation.

Tukey-Lambda Distribution

The fourth histogram is a sample from a Tukey lambda distribution with shape parameter α = 1.2. The standard deviation is 0.49, the median absolute deviation is 0.427, and the range is 1.666.

The Tukey lambda distribution has a range limited to $(-1/\lambda, 1/\lambda)$. That is, it has truncated tails. In this case the standard deviation and median absolute deviation have closer values than for the other three examples which have significant tails.

Robustness

Tukey and Mosteller defined two types of robustness where robustness is a lack of susceptibility to the effects of nonnormality.

1. Robustness of validity means that the confidence intervals for a measure of the population spread (e.g., the standard deviation) have a 95 % chance of covering the true value (i.e., the population value) of that measure of spread regardless of the underlying distribution.

2. Robustness of efficiency refers to high effectiveness in the face of non-normal tails. That is, confidence intervals for the measure of spread tend to be almost as narrow as the best that could be done if we knew the true shape of the distribution.

The standard deviation is an example of an estimator that is the best we can do if the underlying distribution is normal. However, it lacks robustness of validity. That is, confidence intervals based on the standard deviation tend to lack precision if the underlying distribution is in fact not normal.

The median absolute deviation and the interquartile range are estimates of scale that have robustness of validity. However, they are not particularly strong for robustness of efficiency.

If histograms and probability plots indicate that your data are in fact reasonably approximated by a normal distribution, then it makes sense to use the standard deviation as the estimate of scale. However, if your data are not normal, and in particular if there are long tails, then using an alternative measure such as the median absolute deviation, average absolute deviation, or interquartile range makes sense. The range is used in some applications, such as quality control, for its simplicity. In addition, comparing the range to the standard deviation gives an indication of the spread of the data in the tails.

Since the range is determined by the two most extreme points in the data set, we should be cautious about its use for large values of N.

Tukey and Mosteller give a scale estimator that has both robustness of validity and robustness of efficiency. However, it is more complicated and we do not give the formula here.

Software

Most general purpose statistical software programs can generate at least some of the measures of scale discusssed above.

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.3.5.7. Bartlett's Test

Purpose: Test for Homogeneity of Variances

Bartlett's test (Snedecor and Cochran, 1983) is used to test if *k* samples have equal variances. Equal variances across samples is called homogeneity of variances. Some statistical tests, for example the analysis of variance, assume that variances are equal across groups or samples. The Bartlett test can be used to verify that assumption.

Bartlett's test is sensitive to departures from normality. That is, if your samples come from non-normal distributions, then Bartlett's test may simply be testing for non-normality. The Levene test is an alternative to the Bartlett test that is less sensitive to departures from normality.

Definition

The Bartlett test is defined as:

H$_0$: $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$

H$_a$: $\sigma_i^2 \neq \sigma_j^2$ for at least one pair (i,j).

Test Statistic: The Bartlett test statistic is designed to test for equality of variances across groups against the alternative that variances are unequal for at least two groups.

$$T = \frac{(N-k)\ln s_p^2 - \sum_{i=1}^{k}(N_i - 1)\ln s_i^2}{1 + (1/(3(k-1)))((\sum_{i=1}^{k} 1/(N_i - 1)) - 1/(N-k))}$$

In the above, s_i^2 is the variance of the ith group, N is the total sample size, N_i is the sample size of the ith group, k is the number of groups, and s_p^2 is the pooled variance. The pooled variance is a weighted average of the group variances and is defined as:

$$s_p^2 = \sum_{i=1}^{k}(N_i - 1)s_i^2/(N-k)$$

Significance Level: α

Critical Region: The variances are judged to be unequal if,

$$T > \chi^2_{1-\alpha,\, k-1}$$

where $\chi^2_{1-\alpha,\, k-1}$ is the critical value of the chi-square distribution with k - 1 degrees of freedom and a significance level of α.

An alternate definition (Dixon and Massey, 1969) is based on an approximation to the F distribution. This definition is given in the Product and Process Comparisons chapter (chapter 7).

Example

Bartlett's test was performed for the GEAR.DAT data set. The data set contains 10 measurements of gear diameter for ten different batches for a total of 100 measurements.

H_0: $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_{10}^2$
H_a: At least one σ_i^2 is not equal to the others.

Test statistic: T = 20.78580
Degrees of freedom: k - 1 = 9
Significance level: α = 0.05
Critical value: $\chi^2_{1-\alpha,k-1}$ = 16.919
Critical region: Reject H_0 if T > 16.919

We are testing the null hypothesis that the batch variances are all equal. Because the test statistic is larger than the critical value, we reject the null hypotheses at the 0.05 significance level and conclude that at least one batch variance is different from the others.

Question

Bartlett's test can be used to answer the following question:

- Is the assumption of equal variances valid?

Importance

Bartlett's test is useful whenever the assumption of equal variances is made. In particular, this assumption is made for the frequently used one-way analysis of variance. In this case, Bartlett's or Levene's test should be applied to verify the assumption.

Related Techniques

Standard Deviation Plot
Box Plot
Levene Test
Chi-Square Test
Analysis of Variance

Case Study

Heat flow meter data

Software

The Bartlett test is available in many general purpose statistical software programs. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.8. Chi-Square Test for the Variance

Purpose: Test if the variance is equal to a specified value

A chi-square test (Snedecor and Cochran, 1983) can be used to test if the variance of a population is equal to a specified value. This test can be either a two-sided test or a one-sided test. The two-sided version tests against the alternative that the true variance is either less than or greater than the specified value. The one-sided version only tests in one direction. The choice of a two-sided or one-sided test is determined by the problem. For example, if we are testing a new process, we may only be concerned if its variability is greater than the variability of the current process.

Definition

The chi-square hypothesis test is defined as:

H_0: $\sigma^2 = \sigma_0^2$

H_a: $\sigma^2 < \sigma_0^2$ for a lower one-tailed test

$\sigma^2 > \sigma_0^2$ for an upper one-tailed test

$\sigma^2 \neq \sigma_0^2$ for a two-tailed test

Test Statistic: $T = (N-1)/(s/\sigma_0)^2$

where N is the sample size and s is the sample standard deviation. The key element of this formula is the ratio s/σ_0 which compares the ratio of the sample standard deviation to the target standard deviation. The more this ratio deviates from 1, the more likely we are to reject the null hypothesis.

Significance Level: α.

Critical Region: Reject the null hypothesis that the variance is a specified value, σ_0^2, if

$$
\begin{array}{lll}
T > & \chi^2_{1-\alpha,\,N-1} & \text{for an upper one-tailed alternative} \\
T < & \chi^2_{\alpha,\,N-1} & \text{for a lower one-tailed alternative} \\
T < & \chi^2_{\alpha/2,\,N-1} & \text{for a two-tailed test} \\
\text{or} & & \\
T > & \chi^2_{1-\alpha/2,\,N-1} &
\end{array}
$$

where $\chi^2_{.,\,N-1}$ is the critical value of the chi-square distribution with N - 1 degrees of freedom.

The formula for the hypothesis test can easily be converted to form an interval estimate for the variance:

$$\frac{(N-1)s^2}{\chi^2_{1-\alpha/2,\,N-1}} \le \sigma^2 \le \frac{(N-1)s^2}{\chi^2_{\alpha/2,\,N-1}}$$

A confidence interval for the standard deviation is computed by taking the square root of the upper and lower limits of the confidence interval for the variance.

Chi-Square Test Example

A chi-square test was performed for the GEAR.DAT data set. The observed variance for the 100 measurements of gear diameter is 0.00003969 (the standard deviation is 0.0063). We will test the null hypothesis that the true variance is equal to 0.01.

H_0: σ^2 = 0.01
H_a: $\sigma^2 \ne$ 0.01

```
Test statistic:   T = 0.3903
Degrees of freedom:   N - 1 = 99
Significance level:   α = 0.05
Critical values:  χ²_{α/2,N-1} = 73.361
                  χ²_{1-α/2,N-1} = 128.422
Critical region: Reject H0 if T < 73.361 or T > 128.422
```

The test statistic value of 0.3903 is much smaller than the lower critical value, so we reject the null hypothesis and conclude that the variance is not equal to 0.01.

Questions

The chi-square test can be used to answer the following questions:

1. Is the variance equal to some pre-determined threshold value?
2. Is the variance greater than some pre-determined threshold value?
3. Is the variance less than some pre-determined threshold value?

Related Techniques

F Test
Bartlett Test
Levene Test

Software

The chi-square test for the variance is available in many general purpose statistical software programs. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.8.1. Data Used for Chi-Square Test for the Variance

Data Used for Chi-Square Test for the Variance Example

The following are the data used for the chi-square test for the variance example. The first column is gear diameter and the second column is batch number. Only the first column is used for this example.

```
1.006         1.000
0.996         1.000
0.998         1.000
1.000         1.000
0.992         1.000
0.993         1.000
1.002         1.000
0.999         1.000
0.994         1.000
1.000         1.000
0.998         2.000
1.006         2.000
1.000         2.000
1.002         2.000
0.997         2.000
0.998         2.000
0.996         2.000
1.000         2.000
1.006         2.000
0.988         2.000
0.991         3.000
0.987         3.000
0.997         3.000
0.999         3.000
0.995         3.000
0.994         3.000
1.000         3.000
0.999         3.000
0.996         3.000
0.996         3.000
1.005         4.000
1.002         4.000
0.994         4.000
1.000         4.000
0.995         4.000
0.994         4.000
0.998         4.000
0.996         4.000
1.002         4.000
0.996         4.000
0.998         5.000
0.998         5.000
0.982         5.000
0.990         5.000
1.002         5.000
0.984         5.000
0.996         5.000
0.993         5.000
0.980         5.000
0.996         5.000
1.009         6.000
1.013         6.000
```

1.009	6.000
0.997	6.000
0.988	6.000
1.002	6.000
0.995	6.000
0.998	6.000
0.981	6.000
0.996	6.000
0.990	7.000
1.004	7.000
0.996	7.000
1.001	7.000
0.998	7.000
1.000	7.000
1.018	7.000
1.010	7.000
0.996	7.000
1.002	7.000
0.998	8.000
1.000	8.000
1.006	8.000
1.000	8.000
1.002	8.000
0.996	8.000
0.998	8.000
0.996	8.000
1.002	8.000
1.006	8.000
1.002	9.000
0.998	9.000
0.996	9.000
0.995	9.000
0.996	9.000
1.004	9.000
1.004	9.000
0.998	9.000
0.999	9.000
0.991	9.000
0.991	10.000
0.995	10.000
0.984	10.000
0.994	10.000
0.997	10.000
0.997	10.000
0.991	10.000
0.998	10.000
1.004	10.000
0.997	10.000

1.3.5.9. *F*-Test for Equality of Two Variances

Purpose: Test if variances from two populations are equal

An *F*-test (Snedecor and Cochran, 1983) is used to test if the variances of two populations are equal. This test can be a two-tailed test or a one-tailed test. The two-tailed version tests against the alternative that the variances are not equal. The one-tailed version only tests in one direction, that is the variance from the first population is either greater than or less than (but not both) the second population variance. The choice is determined by the problem. For example, if we are testing a new process, we may only be interested in knowing if the new process is less variable than the old process.

Definition

The *F* hypothesis test is defined as:

H$_0$: $\sigma_1^2 = \sigma_2^2$

H$_a$: $\sigma_1^2 < \sigma_2^2$ for a lower one-tailed test

$\sigma_1^2 > \sigma_2^2$ for an upper one-tailed test

$\sigma_1^2 \neq \sigma_2^2$ for a two-tailed test

Test Statistic: $F = s_1^2 / s_2^2$

where s_1^2 and s_2^2 are the sample variances. The more this ratio deviates from 1, the stronger the evidence for unequal population variances.

Significance Level: α

Critical Region: The hypothesis that the two variances are equal is rejected if

$F > F_{\alpha, N_1-1, N_2-1}$ for an upper one-tailed test

$F < F_{1-\alpha, N_1-1, N_2-1}$ for a lower one-tailed test

$F < F_{1-\alpha/2, N_1-1, N_2-1}$ for a two-tailed test or

$$F > F_{\alpha/2,\, N_1-1,\, N_2-1}$$

where $F_{\alpha,\, N_1-1,\, N_2-1}$ is the critical value of the F distribution with N_1-1 and N_2-1 degrees of freedom and a significance level of α.

In the above formulas for the critical regions, the Handbook follows the convention that F_α is the upper critical value from the F distribution and $F_{1-\alpha}$ is the lower critical value from the F distribution. Note that this is the opposite of the designation used by some texts and software programs.

F Test Example

The following F-test was generated for the AUTO83B.DAT data set. The data set contains 480 ceramic strength measurements for two batches of material. The summary statistics for each batch are shown below.

```
BATCH 1:
   NUMBER OF OBSERVATIONS      =      240
   MEAN                         =    688.9987
   STANDARD DEVIATION           =    65.54909

BATCH 2:
   NUMBER OF OBSERVATIONS      =      240
   MEAN                         =    611.1559
   STANDARD DEVIATION           =    61.85425
```

We are testing the null hypothesis that the variances for the two batches are equal.

H_0: $\sigma_1^2 = \sigma_2^2$
H_a: $\sigma_1^2 \neq \sigma_2^2$

Test statistic: F = 1.123037
Numerator degrees of freedom: N_1 - 1 = 239
Denominator degrees of freedom: N_2 - 1 = 239
Significance level: α = 0.05
Critical values: $F(1-\alpha/2, N_1-1, N_2-1)$ = 0.7756
$F(\alpha/2, N_1-1, N_2-1)$ = 1.2894
Rejection region: Reject H_0 if F < 0.7756 or F > 1.2894

The F test indicates that there is not enough evidence to reject the null hypothesis that the two batch variancess are equal at the 0.05 significance level.

Questions

The F-test can be used to answer the following questions:

1. Do two samples come from populations with equal variancess?
2. Does a new process, treatment, or test reduce the variability of the current process?

Related Techniques	Quantile-Quantile Plot Bihistogram Chi-Square Test Bartlett's Test Levene Test
Case Study	Ceramic strength data.
Software	The *F*-test for equality of two variances is available in many general purpose statistical software programs. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.10. Levene Test for Equality of Variances

Purpose: Test for Homogeneity of Variances

Levene's test (Levene 1960) is used to test if k samples have equal variances. Equal variances across samples is called homogeneity of variance. Some statistical tests, for example the analysis of variance, assume that variances are equal across groups or samples. The Levene test can be used to verify that assumption.

Levene's test is an alternative to the Bartlett test. The Levene test is less sensitive than the Bartlett test to departures from normality. If you have strong evidence that your data do in fact come from a normal, or nearly normal, distribution, then Bartlett's test has better performance.

Definition

The Levene test is defined as:

H_0:	$\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$
H_a:	$\sigma_i^2 \neq \sigma_j^2$ for at least one pair (i,j).
Test Statistic:	Given a variable Y with sample of size N divided into k subgroups, where N_i is the sample size of the ith subgroup, the Levene test statistic is defined as:

$$W = \frac{(N-k)}{(k-1)} \frac{\sum_{i=1}^{k} N_i (\bar{Z}_{i.} - \bar{Z}_{..})^2}{\sum_{i=1}^{k} \sum_{j=1}^{N_i} (Z_{ij} - \bar{Z}_{i.})^2}$$

where Z_{ij} can have one of the following three definitions:

1. $Z_{ij} = |Y_{ij} - \bar{Y}_{i.}|$

   where $\bar{Y}_{i.}$ is the mean of the ith subgroup.

2. $Z_{ij} = |Y_{ij} - \tilde{Y}_{i.}|$

   where $\tilde{Y}_{i.}$ is the median of the ith subgroup.

3. $Z_{ij} = |Y_{ij} - \bar{Y}'_{i.}|$

where $\bar{Y}'_{i.}$ is the 10% trimmed mean of the ith subgroup.

$\bar{Z}_{i.}$ are the group means of the Z_{ij} and $\bar{Z}_{..}$ is the overall mean of the Z_{ij}.

The three choices for defining Z_{ij} determine the robustness and power of Levene's test. By robustness, we mean the ability of the test to not falsely detect unequal variances when the underlying data are not normally distributed and the variables are in fact equal. By power, we mean the ability of the test to detect unequal variances when the variances are in fact unequal.

Levene's original paper only proposed using the mean. Brown and Forsythe (1974)) extended Levene's test to use either the median or the trimmed mean in addition to the mean. They performed Monte Carlo studies that indicated that using the trimmed mean performed best when the underlying data followed a Cauchy distribution (i.e., heavy-tailed) and the median performed best when the underlying data followed a χ^2_4 (i.e., skewed) distribution. Using the mean provided the best power for symmetric, moderate-tailed, distributions.

Although the optimal choice depends on the underlying distribution, the definition based on the median is recommended as the choice that provides good robustness against many types of non-normal data while retaining good power. If you have knowledge of the underlying distribution of the data, this may indicate using one of the other choices.

Significance Level:	α
Critical Region:	The Levene test rejects the hypothesis that the variances are equal if

$$W > F_{\alpha,\, k-1,\, N-k}$$

where $F_{\alpha,\, k-1,\, N-k}$ is the upper critical value of the F distribution with k-1 and N-k degrees of freedom at a significance level of α.

In the above formulas for the critical regions, the Handbook follows the convention that F_α is the upper critical value from the F distribution and $F_{1-\alpha}$ is the lower critical value. Note that this is the opposite of some texts and software programs.

Levene's Test Example

Levene's test, based on the median, was performed for the GEAR.DAT data set. The data set includes ten measurements of gear diameter for each of ten batches for a total of 100 measurements.

```
H0:  σ1^2 = ... = σ10^2
Ha:  σ1^2 ≠ ... ≠ σ10^2


Test statistic:  W = 1.705910
Degrees of freedom:  k-1 = 10-1 = 9
                     N-k = 100-10 = 90
Significance level:  α = 0.05
Critical value (upper tail):  Fα,k-1,N-k = 1.9855
Critical region: Reject H0 if F > 1.9855
```

We are testing the hypothesis that the group variances are equal. We fail to reject the null hypothesis at the 0.05 significance level since the value of the Levene test statistic is less than the critical value. We conclude that there is insufficient evidence to claim that the variances are not equal.

Question

Levene's test can be used to answer the following question:

- Is the assumption of equal variances valid?

Related Techniques

Standard Deviation Plot
Box Plot
Bartlett Test
Chi-Square Test
Analysis of Variance

Software

The Levene test is available in some general purpose statistical software programs. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.11. Measures of Skewness and Kurtosis

Skewness and Kurtosis

A fundamental task in many statistical analyses is to characterize the *location* and *variability* of a data set. A further characterization of the data includes skewness and kurtosis.

Skewness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point.

Kurtosis is a measure of whether the data are peaked or flat relative to a normal distribution. That is, data sets with high kurtosis tend to have a distinct peak near the mean, decline rather rapidly, and have heavy tails. Data sets with low kurtosis tend to have a flat top near the mean rather than a sharp peak. A uniform distribution would be the extreme case.

The histogram is an effective graphical technique for showing both the skewness and kurtosis of data set.

Definition of Skewness

For univariate data Y_1, Y_2, ..., Y_N, the formula for skewness is:

$$skewness = \frac{\sum_{i=1}^{N}(Y_i - \bar{Y})^3}{(N-1)s^3}$$

where $\bar{Y}$ is the mean, s is the standard deviation, and N is the number of data points. The skewness for a normal distribution is zero, and any symmetric data should have a skewness near zero. Negative values for the skewness indicate data that are skewed left and positive values for the skewness indicate data that are skewed right. By skewed left, we mean that the left tail is long relative to the right tail. Similarly, skewed right means that the right tail is long relative to the left tail. Some measurements have a lower bound and are skewed right. For example, in reliability studies, failure times cannot be negative.

Definition of Kurtosis

For univariate data Y_1, Y_2, ..., Y_N, the formula for kurtosis is:

$$kurtosis = \frac{\sum_{i=1}^{N}(Y_i - \bar{Y})^4}{(N-1)s^4}$$

where $\bar{Y}$ is the mean, s is the standard deviation, and N is the number of data points.

Alternative Definition of Kurtosis

The kurtosis for a standard normal distribution is three. For this reason, some sources use the following definition of kurtosis (often referred to as "excess kurtosis"):

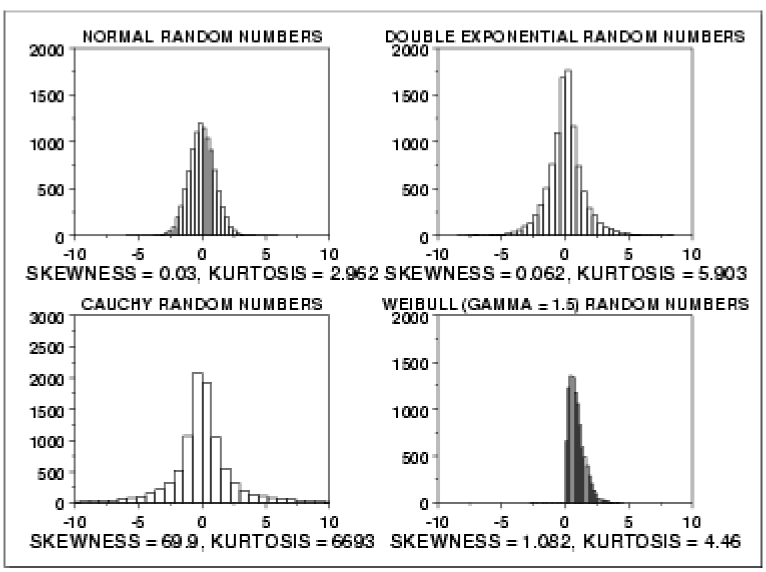
$$kurtosis = \frac{\sum_{i=1}^{N}(Y_i - \bar{Y})^4}{(N-1)s^4} - 3$$

This definition is used so that the standard normal distribution has a kurtosis of zero. In addition, with the second definition positive kurtosis indicates a "peaked" distribution and negative kurtosis indicates a "flat" distribution.

Which definition of kurtosis is used is a matter of convention (this handbook uses the original definition). When using software to compute the sample kurtosis, you need to be aware of which convention is being followed. Many sources use the term kurtosis when they are actually computing "excess kurtosis", so it may not always be clear.

Examples

The following example shows histograms for 10,000 random numbers generated from a normal, a double exponential, a Cauchy, and a Weibull distribution.



Normal Distribution

The first histogram is a sample from a normal distribution. The normal distribution is a symmetric distribution with well-behaved tails. This is indicated by the skewness of 0.03. The kurtosis of 2.96 is near the expected value of 3. The histogram verifies the symmetry.

Double Exponential Distribution

The second histogram is a sample from a double exponential distribution. The double exponential is a symmetric distribution. Compared to the normal, it has a stronger peak, more rapid decay, and heavier tails. That is, we would expect a skewness near zero and a kurtosis higher than 3. The skewness is 0.06 and the kurtosis is 5.9.

Cauchy Distribution

The third histogram is a sample from a Cauchy distribution.

For better visual comparison with the other data sets, we restricted the histogram of the Cauchy distribution to values between -10 and 10. The full data set for the Cauchy data in fact has a minimum of approximately -29,000 and a maximum of approximately 89,000.

The Cauchy distribution is a symmetric distribution with heavy tails and a single peak at the center of the distribution. Since it is symmetric, we would expect a skewness near zero. Due to the heavier tails, we might expect the kurtosis to be larger than for a normal distribution. In fact the skewness is 69.99 and the kurtosis is 6,693. These extremely high values can be explained by the heavy tails. Just as the mean and standard deviation can be distorted by extreme values in the tails, so too can the skewness and kurtosis measures.

Weibull Distribution

The fourth histogram is a sample from a Weibull distribution with shape parameter 1.5. The Weibull distribution is a skewed distribution with the amount of skewness depending on the value of the shape parameter. The degree of decay as we move away from the center also depends on the value of the shape parameter. For this data set, the skewness is 1.08 and the kurtosis is 4.46, which indicates moderate skewness and kurtosis.

Dealing with Skewness and Kurtosis

Many classical statistical tests and intervals depend on normality assumptions. Significant skewness and kurtosis clearly indicate that data are not normal. If a data set exhibits significant skewness or kurtosis (as indicated by a histogram or the numerical measures), what can we do about it?

One approach is to apply some type of transformation to try to make the data normal, or more nearly normal. The Box-Cox transformation is a useful technique for trying to normalize a data set. In particular, taking the log or square root of a data set is often useful for data that exhibit moderate right skewness.

Another approach is to use techniques based on distributions other than the normal. For example, in reliability studies, the exponential, Weibull, and lognormal distributions are typically used as a basis for modeling rather than using the normal distribution. The probability plot correlation coefficient plot and the probability plot are useful tools for determining a good distributional model for the data.

Software The skewness and kurtosis coefficients are available in most general purpose statistical software programs.

1.3.5.12. Autocorrelation

Purpose: Detect Non-Randomness, Time Series Modeling

The autocorrelation (Box and Jenkins, 1976) function can be used for the following two purposes:

1. To detect non-randomness in data.
2. To identify an appropriate time series model if the data are not random.

Definition

Given measurements, $Y_1, Y_2, ..., Y_N$ at time $X_1, X_2, ..., X_N$, the lag k autocorrelation function is defined as

$$r_k = \frac{\sum_{i=1}^{N-k}(Y_i - \bar{Y})(Y_{i+k} - \bar{Y})}{\sum_{i=1}^{N}(Y_i - \bar{Y})^2}$$

Although the time variable, X, is not used in the formula for autocorrelation, the assumption is that the observations are equi-spaced.

Autocorrelation is a correlation coefficient. However, instead of correlation between two different variables, the correlation is between two values of the same variable at times X_i and X_{i+k}.

When the autocorrelation is used to detect non-randomness, it is usually only the first (lag 1) autocorrelation that is of interest. When the autocorrelation is used to identify an appropriate time series model, the autocorrelations are usually plotted for many lags.

Autocorrelation Example

Lag-one autocorrelations were computed for the the LEW.DAT data set.

```
 lag      autocorrelation
  0.         1.00
  1.        -0.31
  2.        -0.74
  3.         0.77
  4.         0.21
  5.        -0.90
  6.         0.38
  7.         0.63
  8.        -0.77
  9.        -0.12
 10.         0.82
 11.        -0.40
 12.        -0.55
 13.         0.73
 14.         0.07
 15.        -0.76
```

```
 16.      0.40
 17.      0.48
 18.     -0.70
 19.     -0.03
 20.      0.70
 21.     -0.41
 22.     -0.43
 23.      0.67
 24.      0.00
 25.     -0.66
 26.      0.42
 27.      0.39
 28.     -0.65
 29.      0.03
 30.      0.63
 31.     -0.42
 32.     -0.36
 33.      0.64
 34.     -0.05
 35.     -0.60
 36.      0.43
 37.      0.32
 38.     -0.64
 39.      0.08
 40.      0.58
 41.     -0.45
 42.     -0.28
 43.      0.62
 44.     -0.10
 45.     -0.55
 46.      0.45
 47.      0.25
 48.     -0.61
 49.      0.14
```

Questions

The autocorrelation function can be used to answer the following questions.

1. Was this sample data set generated from a random process?
2. Would a non-linear or time series model be a more appropriate model for these data than a simple constant plus error model?

Importance

Randomness is one of the key assumptions in determining if a univariate statistical process is in control. If the assumptions of constant location and scale, randomness, and fixed distribution are reasonable, then the univariate process can be modeled as:

$$Y_i = A_0 + E_i$$

where E_i is an error term.

If the randomness assumption is not valid, then a different model needs to be used. This will typically be either a time series model or a non-linear model (with time as the independent variable).

Related Techniques

Autocorrelation Plot
Run Sequence Plot
Lag Plot
Runs Test

Case Study

The heat flow meter data demonstrate the use of autocorrelation in determining if the data are from a random process.

Software

The autocorrelation capability is available in most general purpose statistical software programs. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.13. Runs Test for Detecting Non-randomness

Purpose: Detect Non-Randomness

The runs test (Bradley, 1968) can be used to decide if a data set is from a random process.

A run is defined as a series of increasing values or a series of decreasing values. The number of increasing, or decreasing, values is the length of the run. In a random data set, the probability that the (I+1)th value is larger or smaller than the Ith value follows a binomial distribution, which forms the basis of the runs test.

Typical Analysis and Test Statistics

The first step in the runs test is to count the number of runs in the data sequence. There are several ways to define runs in the literature, however, in all cases the formulation must produce a dichotomous sequence of values. For example, a series of 20 coin tosses might produce the following sequence of heads (H) and tails (T).

H H T T H T H H H H T H H T T T T T H H

The number of runs for this series is nine. There are 11 heads and 9 tails in the sequence.

Definition

We will code values above the median as positive and values below the median as negative. A run is defined as a series of consecutive positive (or negative) values. The runs test is defined as:

H_0:	the sequence was produced in a random manner
H_a:	the sequence was not produced in a random manner
Test Statistic:	The test statistic is

$$Z = \frac{R - \bar{R}}{s_R}$$

where R is the observed number of runs, $\bar{R}$, is the expected number of runs, and s_R is the standard deviation of the number of runs. The values of $\bar{R}$

and s_R are computed as follows:

$$\bar{R} = \frac{2n_1n_2}{n_1+n_2} + 1$$

$$s_R^2 = \frac{2n_1n_2(2n_1n_2 - n_1 - n_2)}{(n_1+n_2)^2(n_1+n_2-1)}$$

where n_1 and n_2 are the number of positive and negative values in the series.

Significance Level: α

Critical Region: The runs test rejects the null hypothesis if

$|Z| > Z_{1-\alpha/2}$

For a large-sample runs test (where $n_1 > 10$ and $n_2 > 10$), the test statistic is compared to a standard normal table. That is, at the 5 % significance level, a test statistic with an absolute value greater than 1.96 indicates non-randomness. For a small-sample runs test, there are tables to determine critical values that depend on values of n_1 and n_2 (Mendenhall, 1982).

Runs Test Example

A runs test was performed for 200 measurements of beam deflection contained in the LEW.DAT data set.

```
H0:  the sequence was produced in a random manner
Ha:  the sequence was not produced in a random
manner
```

```
Test statistic:  Z = 2.6938
Significance level:  α = 0.05
Critical value (upper tail):  Z1-α/2 = 1.96
Critical region: Reject H0 if |Z| > 1.96
```

Since the test statistic is greater than the critical value, we conclude that the data are not random at the 0.05 significance level.

Question

The runs test can be used to answer the following question:

- Were these sample data generated from a random process?

Importance

Randomness is one of the key assumptions in determining if a univariate statistical process is in control. If the assumptions of constant location and scale, randomness, and fixed distribution are reasonable, then the univariate process can be modeled as:

$$Y_i = A_0 + E_i$$

where E_i is an error term.

If the randomness assumption is not valid, then a different model needs to be used. This will typically be either a times series model or a non-linear model (with time as the independent variable).

Related Techniques

Autocorrelation
Run Sequence Plot
Lag Plot

Case Study

Heat flow meter data

Software

Most general purpose statistical software programs support a runs test. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.14. Anderson-Darling Test

Purpose: Test for Distributional Adequacy

The Anderson-Darling test (Stephens, 1974) is used to test if a sample of data came from a population with a specific distribution. It is a modification of the Kolmogorov-Smirnov (K-S) test and gives more weight to the tails than does the K-S test. The K-S test is distribution free in the sense that the critical values do not depend on the specific distribution being tested. The Anderson-Darling test makes use of the specific distribution in calculating critical values. This has the advantage of allowing a more sensitive test and the disadvantage that critical values must be calculated for each distribution. Currently, tables of critical values are available for the normal, lognormal, exponential, Weibull, extreme value type I, and logistic distributions. We do not provide the tables of critical values in this Handbook (see Stephens 1974, 1976, 1977, and 1979) since this test is usually applied with a statistical software program that will print the relevant critical values.

The Anderson-Darling test is an alternative to the chi-square and Kolmogorov-Smirnov goodness-of-fit tests.

Definition

The Anderson-Darling test is defined as:

H_0: The data follow a specified distribution.

H_a: The data do not follow the specified distribution

Test Statistic: The Anderson-Darling test statistic is defined as

$$A^2 = -N - S$$

where

$$S = \sum_{i=1}^{N} \frac{(2i-1)}{N} \left[\ln F(Y_i) + \ln\left(1 - F(Y_{N+1-i})\right) \right]$$

F is the cumulative distribution function of the specified distribution. Note that the Y_i are the *ordered* data.

Significance Level: α

Critical Region: The critical values for the Anderson-Darling test are dependent on the specific distribution that is being tested. Tabulated values and formulas have been published (Stephens, 1974, 1976, 1977, 1979) for a few specific distributions (normal, lognormal, exponential, Weibull, logistic, extreme value type 1). The test is a one-sided test and the hypothesis that the distribution is of a specific form is rejected if the test statistic, A, is greater than the critical value.

Note that for a given distribution, the Anderson-Darling statistic may be multiplied by a constant (which usually depends on the sample size, *n*). These constants are given in the various papers by Stephens. In the sample output below, the test statistic values are adjusted. Also, be aware that different constants (and therefore critical values) have been published. You just need to be aware of what constant was used for a given set of critical values (the needed constant is typically given with the critical values).

Sample Output

We generated 1,000 random numbers for normal, double exponential, Cauchy, and lognormal distributions. In all four cases, the Anderson-Darling test was applied to test for a normal distribution.

The normal random numbers were stored in the variable Y1, the double exponential random numbers were stored in the variable Y2, the Cauchy random numbers were stored in the variable Y3, and the lognormal random numbers were stored in the variable Y4.

```
      Distribution                    Mean        Standard 
Deviation
      ------------                  --------      ---------
---------
      Normal (Y1)                   0.004360
1.001816
      Double Exponential (Y2)      0.020349
1.321627
      Cauchy (Y3)                   1.503854
35.130590
      Lognormal (Y4)                1.518372
1.719969

      H0:  the data are normally distributed
      Ha:  the data are not normally distributed

      Y1 adjusted test statistic:  A2 =   0.2576
      Y2 adjusted test statistic:  A2 =   5.8492
      Y3 adjusted test statistic:  A2 = 288.7863
      Y4 adjusted test statistic:  A2 =  83.3935

      Significance level:  α = 0.05
      Critical value:  0.752
      Critical region:  Reject H0 if A2 > 0.752
```

When the data were generated using a normal distribution, the test statistic was small and the hypothesis of normality was not rejected. When the data were generated using the double exponential, Cauchy, and lognormal distributions, the test statistics were large, and the hypothesis of an underlying normal distribution was rejected at the 0.05 significance level.

Questions

The Anderson-Darling test can be used to answer the following questions:

- Are the data from a normal distribution?
- Are the data from a log-normal distribution?
- Are the data from a Weibull distribution?
- Are the data from an exponential distribution?
- Are the data from a logistic distribution?

Importance

Many statistical tests and procedures are based on specific distributional assumptions. The assumption of normality is particularly common in classical statistical tests. Much reliability modeling is based on the assumption that the data follow a Weibull distribution.

There are many non-parametric and robust techniques that do not make strong distributional assumptions. However, techniques based on specific distributional assumptions are in general more powerful than non-parametric and robust techniques. Therefore, if the distributional assumptions can be validated, they are generally preferred.

Related Techniques

Chi-Square goodness-of-fit Test
Kolmogorov-Smirnov Test
Shapiro-Wilk Normality Test
Probability Plot
Probability Plot Correlation Coefficient Plot

Case Study

Josephson junction cryothermometry case study.

Software

The Anderson-Darling goodness-of-fit test is available in some general purpose statistical software programs. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.15. Chi-Square Goodness-of-Fit Test

Purpose: Test for distributional adequacy

The chi-square test (Snedecor and Cochran, 1989) is used to test if a sample of data came from a population with a specific distribution.

An attractive feature of the chi-square goodness-of-fit test is that it can be applied to any univariate distribution for which you can calculate the cumulative distribution function. The chi-square goodness-of-fit test is applied to binned data (i.e., data put into classes). This is actually not a restriction since for non-binned data you can simply calculate a histogram or frequency table before generating the chi-square test. However, the value of the chi-square test statistic are dependent on how the data is binned. Another disadvantage of the chi-square test is that it requires a sufficient sample size in order for the chi-square approximation to be valid.

The chi-square test is an alternative to the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests. The chi-square goodness-of-fit test can be applied to discrete distributions such as the binomial and the Poisson. The Kolmogorov-Smirnov and Anderson-Darling tests are restricted to continuous distributions.

Additional discussion of the chi-square goodness-of-fit test is contained in the product and process comparisons chapter (chapter 7).

Definition

The chi-square test is defined for the hypothesis:

H_0:	The data follow a specified distribution.
H_a:	The data do not follow the specified distribution.
Test Statistic:	For the chi-square goodness-of-fit computation, the data are divided into k bins and the test statistic is defined as

$$\chi^2 = \sum_{i=1}^{k} (O_i - E_i)^2 / E_i$$

where O_i is the observed frequency for bin *i* and E_i is the expected frequency for bin *i*. The expected frequency is calculated by

$$E_i = N(F(Y_u) - F(Y_l))$$

where F is the cumulative Distribution function for the distribution being tested, Y_u is the upper limit for class i, Y_l is the lower limit for class i, and *N* is the sample size.

This test is sensitive to the choice of bins. There is no optimal choice for the bin width (since the optimal bin width depends on the distribution). Most reasonable choices should produce similar, but not identical, results. For the chi-square approximation to be valid, the expected frequency should be at least 5. This test is not valid for small samples, and if some of the counts are less than five, you may need to combine some bins in the tails.

Significance Level: α.

Critical Region: The test statistic follows, approximately, a chi-square distribution with (*k* - *c*) degrees of freedom where *k* is the number of non-empty cells and *c* = the number of estimated parameters (including location and scale parameters and shape parameters) for the distribution + 1. For example, for a 3-parameter Weibull distribution, *c* = 4.

Therefore, the hypothesis that the data are from a population with the specified distribution is rejected if

$$\chi^2 > \chi^2_{1-\alpha,\, k-c}$$

where $\chi^2_{1-\alpha,\, k-c}$ is the chi-square critical value with *k* - *c* degrees of freedom and significance level α.

Chi-Square Test Example

We generated 1,000 random numbers for normal, double exponential, *t* with 3 degrees of freedom, and lognormal distributions. In all cases, a chi-square test with *k* = 32 bins was applied to test for normally distributed data. Because the normal distribution has two parameters, *c* = 2 + 1 = 3

The normal random numbers were stored in the variable Y1, the double exponential random numbers were stored in the variable Y2, the *t* random numbers were stored in the variable Y3, and the lognormal random numbers were stored in the variable Y4.

```
H0:  the data are normally distributed
Ha:  the data are not normally distributed

Y1 Test statistic:  Χ 2 =   32.256
Y2 Test statistic:  Χ 2 =   91.776
Y3 Test statistic:  Χ 2 =  101.488
Y4 Test statistic:  Χ 2 = 1085.104

Significance level:  α = 0.05
Degrees of freedom:  k - c = 32 - 3 = 29
Critical value:  Χ 2 1-α, k-c = 42.557
Critical region: Reject H0 if Χ 2 > 42.557
```

As we would hope, the chi-square test fails to reject the null hypothesis for the normally distributed data set and rejects the null hypothesis for the three non-normal data sets.

Questions

The chi-square test can be used to answer the following types of questions:

- Are the data from a normal distribution?
- Are the data from a log-normal distribution?
- Are the data from a Weibull distribution?
- Are the data from an exponential distribution?
- Are the data from a logistic distribution?
- Are the data from a binomial distribution?

Importance

Many statistical tests and procedures are based on specific distributional assumptions. The assumption of normality is particularly common in classical statistical tests. Much reliability modeling is based on the assumption that the distribution of the data follows a Weibull distribution.

There are many non-parametric and robust techniques that are not based on strong distributional assumptions. By non-parametric, we mean a technique, such as the sign test, that is not based on a specific distributional assumption. By robust, we mean a statistical technique that performs well under a wide range of distributional assumptions. However, techniques based on specific distributional assumptions are in general more powerful than these non-parametric and robust techniques. By power, we mean the ability to detect a difference when that difference actually exists. Therefore, if the distributional assumption can be confirmed, the parametric techniques are generally preferred.

If you are using a technique that makes a normality (or some other type of distributional) assumption, it is important to confirm that this assumption is in fact justified. If it is, the more powerful parametric techniques can be used. If the distributional assumption is not justified, a non-parametric or robust technique may be required.

Related Techniques

Anderson-Darling Goodness-of-Fit Test
Kolmogorov-Smirnov Test
Shapiro-Wilk Normality Test
Probability Plots
Probability Plot Correlation Coefficient Plot

Software

Some general purpose statistical software programs provide a chi-square goodness-of-fit test for at least some of the common distributions. Both Dataplot code and R code can be used to generate the analyses in this section.

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.3.5.16. Kolmogorov-Smirnov Goodness-of-Fit Test

Purpose: Test for Distributional Adequacy

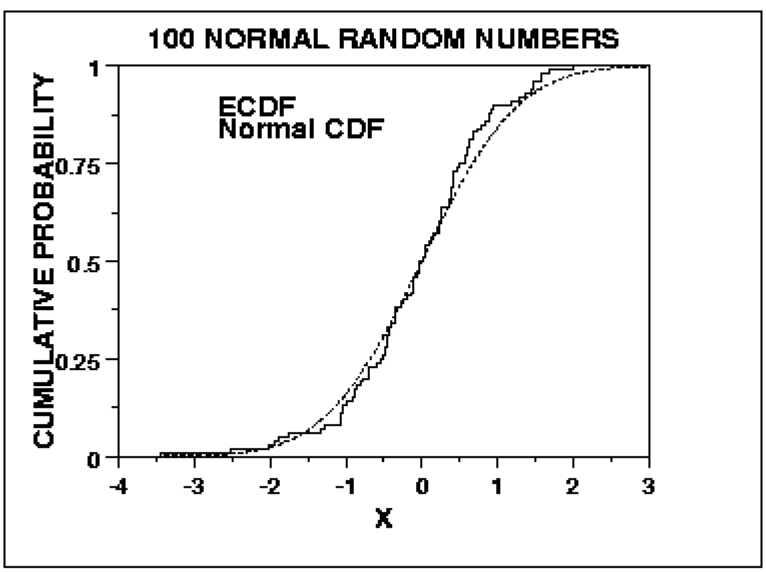
The Kolmogorov-Smirnov test (Chakravart, Laha, and Roy, 1967) is used to decide if a sample comes from a population with a specific distribution.

The Kolmogorov-Smirnov (K-S) test is based on the empirical distribution function (ECDF). Given *N ordered* data points Y_1, Y_2, ..., Y_N, the ECDF is defined as

$$E_N = n(i)/N$$

where $n(i)$ is the number of points less than Y_i and the Y_i are ordered from smallest to largest value. This is a step function that increases by $1/N$ at the value of each ordered data point.

The graph below is a plot of the empirical distribution function with a normal cumulative distribution function for 100 normal random numbers. The K-S test is based on the maximum distance between these two curves.



Characteristics and Limitations of

An attractive feature of this test is that the distribution of the K-S test statistic itself does not depend on the underlying cumulative distribution function being tested. Another advantage is that it is *the K-S Test*

an exact test (the chi-square goodness-of-fit test depends on an adequate sample size for the approximations to be valid). Despite these advantages, the K-S test has several important limitations:

1. It only applies to continuous distributions.
2. It tends to be more sensitive near the center of the distribution than at the tails.
3. Perhaps the most serious limitation is that the distribution must be fully specified. That is, if location, scale, and shape parameters are estimated from the data, the critical region of the K-S test is no longer valid. It typically must be determined by simulation.

Due to limitations 2 and 3 above, many analysts prefer to use the Anderson-Darling goodness-of-fit test. However, the Anderson-Darling test is only available for a few specific distributions.

Definition

The Kolmogorov-Smirnov test is defined by:

H_0: The data follow a specified distribution

H_a: The data do not follow the specified distribution

Test Statistic: The Kolmogorov-Smirnov test statistic is defined as

$$D = \max_{1 \le i \le N} \left(F(Y_i) - \frac{i-1}{N}, \frac{i}{N} - F(Y_i) \right)$$

where ***F*** is the theoretical cumulative distribution of the distribution being tested which must be a continuous distribution (i.e., no discrete distributions such as the binomial or Poisson), and it must be fully specified (i.e., the location, scale, and shape parameters cannot be estimated from the data).

Significance Level: α.

Critical Values: The hypothesis regarding the distributional form is rejected if the test statistic, ***D***, is greater than the critical value obtained from a table. There are several variations of these tables in the literature that use somewhat different scalings for the K-S test statistic and critical regions. These alternative formulations should be equivalent, but it is necessary to ensure that the test statistic is calculated in a way that is consistent with how the critical values were tabulated.

We do not provide the K-S tables in the Handbook since software programs that perform a K-S test will provide the relevant critical values.

Technical Note

Previous editions of e-Handbook gave the following formula for the computation of the Kolmogorov-Smirnov goodness of fit statistic:

$$D = \max_{1 \le i \le N} \left| F(Y_i) - \frac{i}{N} \right|$$

This formula is in fact not correct. Note that this formula can be rewritten as:

$$D = \max_{1 \le i \le N} \left(F(Y_i) - \frac{i}{N}, \frac{i}{N} - F(Y_i) \right)$$

This form makes it clear that an upper bound on the difference between these two formulas is *i*/*N*. For actual data, the difference is likely to be less than the upper bound.

For example, for *N* = 20, the upper bound on the difference between these two formulas is 0.05 (for comparison, the 5% critical value is 0.294). For *N* = 100, the upper bound is 0.001. In practice, if you have moderate to large sample sizes (say *N* ≥ 50), these formulas are essentially equivalent.

Kolmogorov-Smirnov Test Example

We generated 1,000 random numbers for normal, double exponential, *t* with 3 degrees of freedom, and lognormal distributions. In all cases, the Kolmogorov-Smirnov test was applied to test for a normal distribution.

The normal random numbers were stored in the variable Y1, the double exponential random numbers were stored in the variable Y2, the *t* random numbers were stored in the variable Y3, and the lognormal random numbers were stored in the variable Y4.

H_0: the data are normally distributed
H_a: the data are not normally distributed

Y1 test statistic: D = 0.0241492
Y2 test statistic: D = 0.0514086
Y3 test statistic: D = 0.0611935
Y4 test statistic: D = 0.5354889

Significance level: α = 0.05
Critical value: 0.04301
Critical region: Reject H_0 if D > 0.04301

As expected, the null hypothesis is not rejected for the normally distributed data, but is rejected for the remaining three data sets that are not normally distributed.

Questions

The Kolmogorov-Smirnov test can be used to answer the following types of questions:

- Are the data from a normal distribution?
- Are the data from a log-normal distribution?
- Are the data from a Weibull distribution?
- Are the data from an exponential distribution?
- Are the data from a logistic distribution?

Importance	Many statistical tests and procedures are based on specific distributional assumptions. The assumption of normality is particularly common in classical statistical tests. Much reliability modeling is based on the assumption that the data follow a Weibull distribution. There are many non-parametric and robust techniques that are not based on strong distributional assumptions. By non-parametric, we mean a technique, such as the sign test, that is not based on a specific distributional assumption. By robust, we mean a statistical technique that performs well under a wide range of distributional assumptions. However, techniques based on specific distributional assumptions are in general more powerful than these non-parametric and robust techniques. By power, we mean the ability to detect a difference when that difference actually exists. Therefore, if the distributional assumptions can be confirmed, the parametric techniques are generally preferred. If you are using a technique that makes a normality (or some other type of distributional) assumption, it is important to confirm that this assumption is in fact justified. If it is, the more powerful parametric techniques can be used. If the distributional assumption is not justified, using a non-parametric or robust technique may be required.
Related Techniques	Anderson-Darling goodness-of-fit Test Chi-Square goodness-of-fit Test Shapiro-Wilk Normality Test Probability Plots Probability Plot Correlation Coefficient Plot
Software	Some general purpose statistical software programs support the Kolmogorov-Smirnov goodness-of-fit test, at least for the more common distributions. Both Dataplot code and R code can be used to generate the analyses in this section.

1.3.5.17. Detection of Outliers

Introduction

An outlier is an observation that appears to deviate markedly from other observations in the sample.

Identification of potential outliers is important for the following reasons.

1. An outlier may indicate bad data. For example, the data may have been coded incorrectly or an experiment may not have been run correctly. If it can be determined that an outlying point is in fact erroneous, then the outlying value should be deleted from the analysis (or corrected if possible).

2. In some cases, it may not be possible to determine if an outlying point is bad data. Outliers may be due to random variation or may indicate something scientifically interesting. In any event, we typically do not want to simply delete the outlying observation. However, if the data contains significant outliers, we may need to consider the use of robust statistical techniques.

Labeling, Accomodation, Identification

Iglewicz and Hoaglin distinguish the three following issues with regards to outliers.

1. outlier labeling - flag potential outliers for further investigation (i.e., are the potential outliers erroneous data, indicative of an inappropriate distributional model, and so on).

2. outlier accomodation - use robust statistical techniques that will not be unduly affected by outliers. That is, if we cannot determine that potential outliers are erroneous observations, do we need modify our statistical analysis to more appropriately account for these observations?

3. outlier identification - formally test whether observations are outliers.

This section focuses on the labeling and identification issues.

Normality Assumption

Identifying an observation as an outlier depends on the underlying distribution of the data. In this section, we limit the discussion to univariate data sets that are assumed to follow an approximately normal distribution. If the normality assumption for the data being tested is not valid, then a determination that there is an outlier may in fact be due to the non-normality of the data rather than the prescence of an outlier.

For this reason, it is recommended that you generate a normal probability plot of the data before applying an outlier test. Although you can also perform formal tests for normality, the prescence of one or more outliers may cause the tests to reject normality when it is in fact a reasonable assumption for applying the outlier test.

In addition to checking the normality assumption, the lower and upper tails of the normal probability plot can be a useful graphical technique for identifying potential outliers. In particular, the plot can help determine whether we need to check for a single outlier or whether we need to check for multiple outliers.

The box plot and the histogram can also be useful graphical tools in checking the normality assumption and in identifying potential outliers.

Single Versus Multiple Outliers

Some outlier tests are designed to detect the prescence of a single outlier while other tests are designed to detect the prescence of multiple outliers. It is not appropriate to apply a test for a single outlier sequentially in order to detect multiple outliers.

In addition, some tests that detect multiple outliers may require that you specify the number of suspected outliers exactly.

Masking and Swamping

Masking can occur when we specify too few outliers in the test. For example, if we are testing for a single outlier when there are in fact two (or more) outliers, these additional outliers may influence the value of the test statistic enough so that no points are declared as outliers.

On the other hand, swamping can occur when we specify too many outliers in the test. For example, if we are testing for two or more outliers when there is in fact only a single outlier, both points may be declared outliers (many tests will declare either all or none of the tested points as outliers).

Due to the possibility of masking and swamping, it is useful to complement formal outlier tests with graphical methods. Graphics can often help identify cases where masking or swamping may be an issue. Swamping and masking are also the reason that many tests require that the exact number of outliers being tested must be specified.

Also, masking is one reason that trying to apply a single outlier test sequentially can fail. For example, if there are multiple outliers, masking may cause the outlier test for the first outlier to return a conclusion of no outliers (and so the testing for any additional outliers is not performed).

Z-Scores and Modified Z-Scores

The Z-score of an observation is defined as

$$Z_i = \frac{Y_i - \bar{Y}}{s}$$

with $\bar{Y}$ and *s* denoting the sample mean and sample standard deviation, respectively. In other words, data is given in units of how many standard deviations it is from the mean.

Although it is common practice to use Z-scores to identify possible outliers, this can be misleading (partiucarly for small sample sizes) due to the fact that the maximum Z-score is at most $(n-1)/\sqrt{n}$.

Iglewicz and Hoaglin recommend using the modified Z-score

$$M_i = \frac{0.6745(x_i - \tilde{x})}{\mathrm{MAD}}$$

with MAD denoting the median absolute deviation and $\tilde{x}$ denoting the median.

These authors recommend that modified Z-scores with an absolute value of greater than 3.5 be labeled as potential outliers.

Formal Outlier Tests

A number of formal outlier tests have proposed in the literature. These can be grouped by the following characteristics:

- What is the distributional model for the data? We restrict our discussion to tests that assume the data follow an approximately normal distribution.
- Is the test designed for a single outlier or is it designed for multiple outliers?
- If the test is designed for multiple outliers, does the number of outliers need to be specified exactly or can we specify an upper bound for the number of outliers?

The following are a few of the more commonly used outlier tests for normally distributed data. This list is not exhaustive (a large number of outlier tests have been proposed in the literature). The tests given here are essentially based on the criterion of "distance from the mean". This is not the only criterion that could be used. For example, the Dixon test, which is not discussed here, is based a value being too large (or small) compared to its nearest neighbor.

1. Grubbs' Test - this is the recommended test when testing for a single outlier.
2. Tietjen-Moore Test - this is a generalization of the Grubbs' test to the case of more than one outlier. It has the limitation that the number of outliers must be specified exactly.
3. Generalized Extreme Studentized Deviate (ESD) Test - this test requires only an upper bound on the suspected number of outliers and is the recommended test when the exact number of outliers is not known.

Lognormal Distribution

The tests discussed here are specifically based on the assumption that the data follow an approximately normal disribution. If your data follow an approximately lognormal distribution, you can transform the data to normality by taking the logarithms of the data and then applying the outlier tests discussed here.

Further Information

Iglewicz and Hoaglin provide an extensive discussion of the outlier tests given above (as well as some not given above) and also give a good tutorial on the subject of outliers. Barnett and Lewis provide a book length treatment of the subject.

In addition to discussing additional tests for data that follow an approximately normal distribution, these sources also discuss the case where the data are not normally distributed.

1.3.5.18. Yates Algorithm

Purpose: Estimate Factor Effects in a 2-Level Factorial Design

Full factorial and fractional factorial designs are common in designed experiments for engineering and scientific applications.

In these designs, each factor is assigned two levels. These are typically called the low and high levels. For computational purposes, the factors are scaled so that the low level is assigned a value of -1 and the high level is assigned a value of +1. These are also commonly referred to as "-" and "+".

A full factorial design contains all possible combinations of low/high levels for all the factors. A fractional factorial design contains a carefully chosen subset of these combinations. The criterion for choosing the subsets is discussed in detail in the process improvement chapter.

The Yates algorithm exploits the special structure of these designs to generate least squares estimates for factor effects for all factors and all relevant interactions.

The mathematical details of the Yates algorithm are given in chapter 10 of Box, Hunter, and Hunter (1978). Natrella (1963) also provides a procedure for testing the significance of effect estimates.

The effect estimates are typically complemented by a number of graphical techniques such as the DOE mean plot and the DOE contour plot ("DOE" represents "design of experiments"). These are demonstrated in the eddy current case study.

Yates Order

Before performing the Yates algorithm, the data should be arranged in "Yates order". That is, given k factors, the kth column consists of 2^{k-1} minus signs (i.e., the low level of the factor) followed by 2^{k-1} plus signs (i.e., the high level of the factor). For example, for a full factorial design with three factors, the design matrix is

```
- - -
+ - -
- + -
```

```
+ + -
- - +
+ - +
- + +
+ + +
```

Determining the Yates order for fractional factorial designs requires knowledge of the confounding structure of the fractional factorial design.

Yates Algorithm

The Yates algorithm is demonstrated for the eddy current data set. The data set contains eight measurements from a two-level, full factorial design with three factors. The purpose of the experiment is to identify factors that have the most effect on eddy current measurements.

In the "Effect" column, we list the main effects and interactions from our factorial experiment in standard order. In the "Response" column, we list the measurement results from our experiment in Yates order.

Effect	Response	Col 1	Col 2	Col 3	Estimate
Mean	1.70	6.27	10.21	21.27	2.65875
X1	4.57	3.94	11.06	12.41	1.55125
X2	0.55	6.10	5.71	-3.47	-0.43375
X1*X2	3.39	4.96	6.70	0.51	0.06375
X3	1.51	2.87	-2.33	0.85	0.10625
X1*X3	4.59	2.84	-1.14	0.99	0.12375
X2*X3	0.67	3.08	-0.03	1.19	0.14875
X1*X2*X3	4.29	3.62	0.54	0.57	0.07125

```
Sum of responses:            21.27
Sum-of-squared responses:    77.7707
Sum-of-squared Col 3:       622.1656
```

The first four values in Col 1 are obtained by adding adjacent pairs of responses, for example 4.57 + 1.70 = 6.27, and 3.39 + 0.55 = 3.94. The second four values in Col 1 are obtained by subtracting the same adjacent pairs of responses, for example, 4.57 - 1.70 = 2.87, and 3.39 - 0.55 = 2.84. The values in Col 2 are calculated in the same way, except that we are adding and subtracting adjacent values from Col 1. Col 3 is computed using adjacent values from Col 2. Finally, we obtain the "Estimate" column by dividing the values in Col 3 by the total number of responses, 8.

We can check our calculations by making sure that the first value in Col 3 (21.27) is the sum of all the responses. In addition, the sum-of-squared responses (77.7707) should equal the sum-of-squared Col 3 values divided by 8 (622.1656/8 = 77.7707).

Practical Considerations

The Yates algorithm provides a convenient method for computing effect estimates; however, the same information is easily obtained from statistical software using either an analysis of variance or regression procedure. The methods for analyzing data from a designed experiment are discussed more fully in the chapter on Process Improvement.

Graphical Presentation

The following plots may be useful to complement the quantitative information from the Yates algorithm.

1. Ordered data plot
2. Ordered absolute effects plot
3. Cumulative residual standard deviation plot

Questions

The Yates algorithm can be used to answer the following question.

1. What is the estimated effect of a factor on the response?

Related Techniques

Multi-factor analysis of variance
DOE mean plot
Block plot
DOE contour plot

Case Study

The analysis of a full factorial design is demonstrated in the eddy current case study.

Software

All statistical software packages are capable of estimating effects using an analysis of variance or least squares regression procedure.

1.3.5.18.1. Defining Models and Prediction Equations

For Orthogonal Designs, Parameter Estimates Don't Change as Additional Terms Are Added

In most cases of least-squares fitting, the model coefficients for previously added terms change depending on what was successively added. For example, the X1 coefficient might change depending on whether or not an X2 term was included in the model. This is **not** the case when the design is orthogonal, as is a 2^3 full factorial design. For orthogonal designs, the estimates for the previously included terms do not change as additional terms are added. This means the ranked list of parameter estimates are the least-squares coefficient estimates for progressively more complicated models.

Example Prediction Equation

We use the parameter estimates derived from a least-squares analysis for the eddy current data set to create an example prediction equation.

```
Parameter     Estimate
---------     --------
Mean           2.65875
X1             1.55125
X2            -0.43375
X1*X2          0.06375
X3             0.10625
X1*X3          0.12375
X2*X3          0.14875
X1*X2*X3       0.07125
```

A prediction equation predicts a value of the reponse variable for given values of the factors. The equation we select can include all the factors shown above, or it can include a subset of the factors. For example, one possible prediction equation using only two factors, X1 and X2, is:

$$\hat{Y} = 2.65875 + 1.55125 \cdot X_1 - 0.43375 \cdot X_2$$

The least-squares parameter estimates in the prediction equation reflect the change in response for a one-unit change in the factor value. To obtain "full" effect estimates (as computed using the Yates algorithm) for the change in factor levels from -1 to +1, the effect estimates (except for the intercept) would be multiplied by two.

Remember that the Yates algorithm is just a convenient method for computing effects, any statistical software package with least-squares regression capabilities will produce the same effects as well as many other useful analyses.

Model Selection

We want to select the most appropriate model for our data while balancing the following two goals.

1. We want the model to include all important factors.
2. We want the model to be parsimonious. That is, the model should be as simple as possible.

Note that the residual standard deviation alone is insufficient for determining the most appropriate model as it will always be decreased by adding additional factors. The next section describes a number of approaches for determining which factors (and interactions) to include in the model.

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.3.5.18.2. Important Factors

Identify Important Factors

We want to select the most appropriate model to represent our data. This requires balancing the following two goals.

1. We want the model to include all important factors.
2. We want the model to be parsimonious. That is, the model should be as simple as possible.

In short, we want our model to include all the important factors and interactions and to omit the unimportant factors and interactions.

Seven criteria are utilized to define important factors. These seven criteria are not all equally important, nor will they yield identical subsets, in which case a consensus subset or a weighted consensus subset must be extracted. In practice, some of these criteria may not apply in all situations.

These criteria will be examined in the context of the eddy current data set. The parameter estimates computed using least-squares analysis are shown below.

```
Parameter     Estimate
---------     --------
Mean           2.65875
X1             1.55125
X2            -0.43375
X1*X2          0.06375
X3             0.10625
X1*X3          0.12375
X2*X3          0.14875
X1*X2*X3       0.07125
```

In practice, not all of these criteria will be used with every analysis (and some analysts may have additional criteria). These critierion are given as useful guidelines. Most analysts will focus on those criteria that they find most useful.

Criteria for Including Terms in the Model

The seven criteria that we can use in determining whether to keep a factor in the model can be summarized as follows.

1. Parameters: Engineering Significance
2. Parameters: Order of Magnitude
3. Parameters: Statistical Significance
4. Parameters: Probability Plots
5. Effects: Youden Plot
6. Residual Standard Deviation: Engineering Significance
7. Residual Standard Deviation: Statistical Significance

The first four criteria focus on parameter estimates with three numeric criteria and one graphical criteria. The fifth criteria focuses on effects, which are twice the parameter estimates. The last two criteria focus on the residual standard deviation of the model. We discuss each of these seven criteria in detail in the sections that following.

Parameters: Engineering Significance

The minimum engineering significant difference is defined as

$$|\hat{\beta}_i| > \Delta$$

where $|\hat{\beta}_i|$ is the absolute value of the parameter estimate and Δ is the minimum engineering significant difference.

That is, declare a factor as "important" if the parameter estimate is greater than some a priori declared engineering difference. This implies that the engineering staff have in fact stated what a minimum difference will be. Oftentimes this is not the case. In the absence of an a priori difference, a good rough rule for the minimum engineering significant Δ is to keep only those factors whose parameter estimate is greater than, say, 10% of the current production average. In this case, let's say that the average detector has a sensitivity of 2.5 ohms. This would suggest that we would declare all factors whose parameter is greater than 10 % of 2.5 ohms = 0.25 ohm to be significant (from an engineering point of view).

Based on this minimum engineering significant difference criterion, we conclude that we should keep two terms: X1 and X2.

Parameters: Order of Magnitude

The order of magnitude criterion is defined as

$$|\hat{\beta}_i| < 0.10 * max|\hat{\beta}_i|$$

That is, exclude any factor that is less than 10 % of the maximum parameter size. We may or may not keep the other factors. This criterion is neither engineering nor statistical, but it does offer some additional numerical insight. For the current example, the largest parameter is from X1 (1.55125 ohms), and so 10 % of that is 0.155 ohms, which suggests keeping all factors whose parameters exceed 0.155 ohms.

Based on the order-of-magnitude criterion, we thus conclude that we should keep two terms: X1 and X2. A third term, X2*X3 (0.14875), is just slightly under the cutoff level, so we may consider keeping it based on the other criterion.

Parameters: Statistical Significance

Statistical significance is defined as

$$|\hat{\beta}_i| > 2 \text{ s.e.}(\hat{\beta}_i)$$

That is, declare a factor as important if its parameter is more than 2 standard deviations away from 0 (0, by definition, meaning "no effect").

The "2" comes from normal theory (more specifically, a value of 1.96 yields a 95 % confidence interval). More precise values would come from *t*-distribution theory.

The difficulty with this is that in order to invoke this criterion we need the standard deviation, σ, of an observation. This is problematic because

1. the engineer may not know σ;
2. the experiment might not have replication, and so a model-free estimate of σ is not obtainable;

3. obtaining an estimate of σ by assuming the sometimes- employed assumption of ignoring 3-term interactions and higher may be incorrect from an engineering point of view.

For the eddy current example:

1. the engineer did **not** know σ;
2. the design (a 2^3 full factorial) did **not** have replication;
3. ignoring 3-term interactions and higher interactions leads to an estimate of σ based on omitting only a single term: the X1*X2*X3 interaction.

For the eddy current example, if one assumes that the 3-term interaction is nil and hence represents a single drawing from a population centered at zero, then an estimate of the standard deviation of a parameter is simply the estimate of the 3-factor interaction (0.07125). Two standard deviations is thus 0.1425. For this example, the rule is thus to keep all $|\hat{\beta}_i|$ > 0.1425.

This results in keeping three terms: X1 (1.55125), X2 (-0.43375), and X1*X2 (0.14875).

Parameters: Probability Plots

Probability plots can be used in the following manner.

1. Normal Probability Plot: Keep a factor as "important" if it is well off the line through zero on a normal probability plot of the parameter estimates.

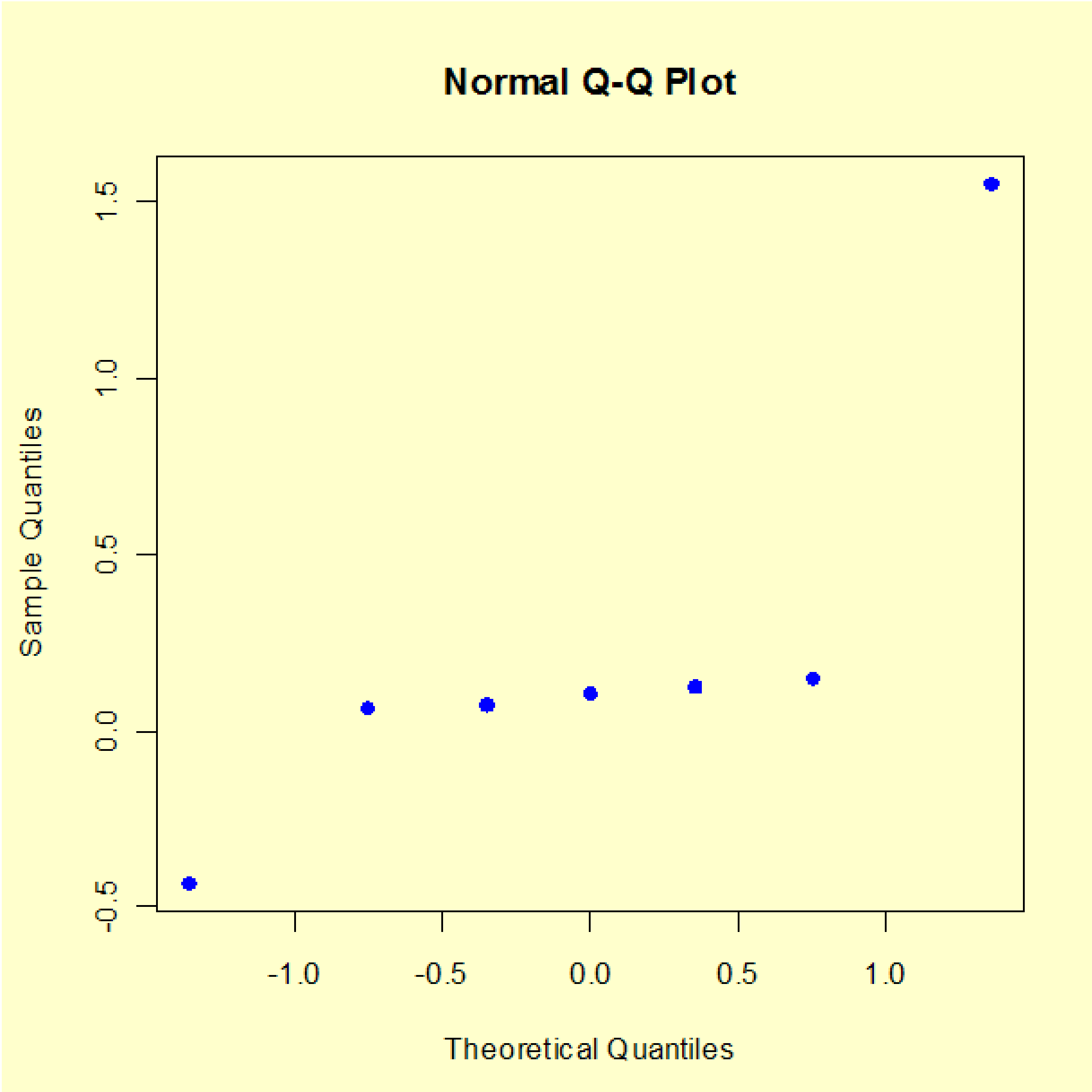
2. Half-Normal Probability Plot: Keep a factor as "important" if it is well off the line near zero on a half-normal probability plot of the absolute value of parameter estimates.

Both of these methods are based on the fact that the least-squares estimates of parameters for these two-level orthogonal designs are simply half the difference of averages and so the central limit theorem, loosely applied, suggests that (if no factor were important) the parameter estimates should have approximately a normal distribution with mean zero and the absolute value of the estimates should have a half-normal distribution.

Since the half-normal probability plot is only concerned with parmeter magnitudes as opposed to signed parameters (which are subject to the vagaries of how the initial factor codings +1 and -1 were assigned), the half-normal probability plot is preferred by some over the normal probability plot.

Normal Probablity Plot of Parameters

The following normal probability plot shows the parameter estimates for the eddy current data.



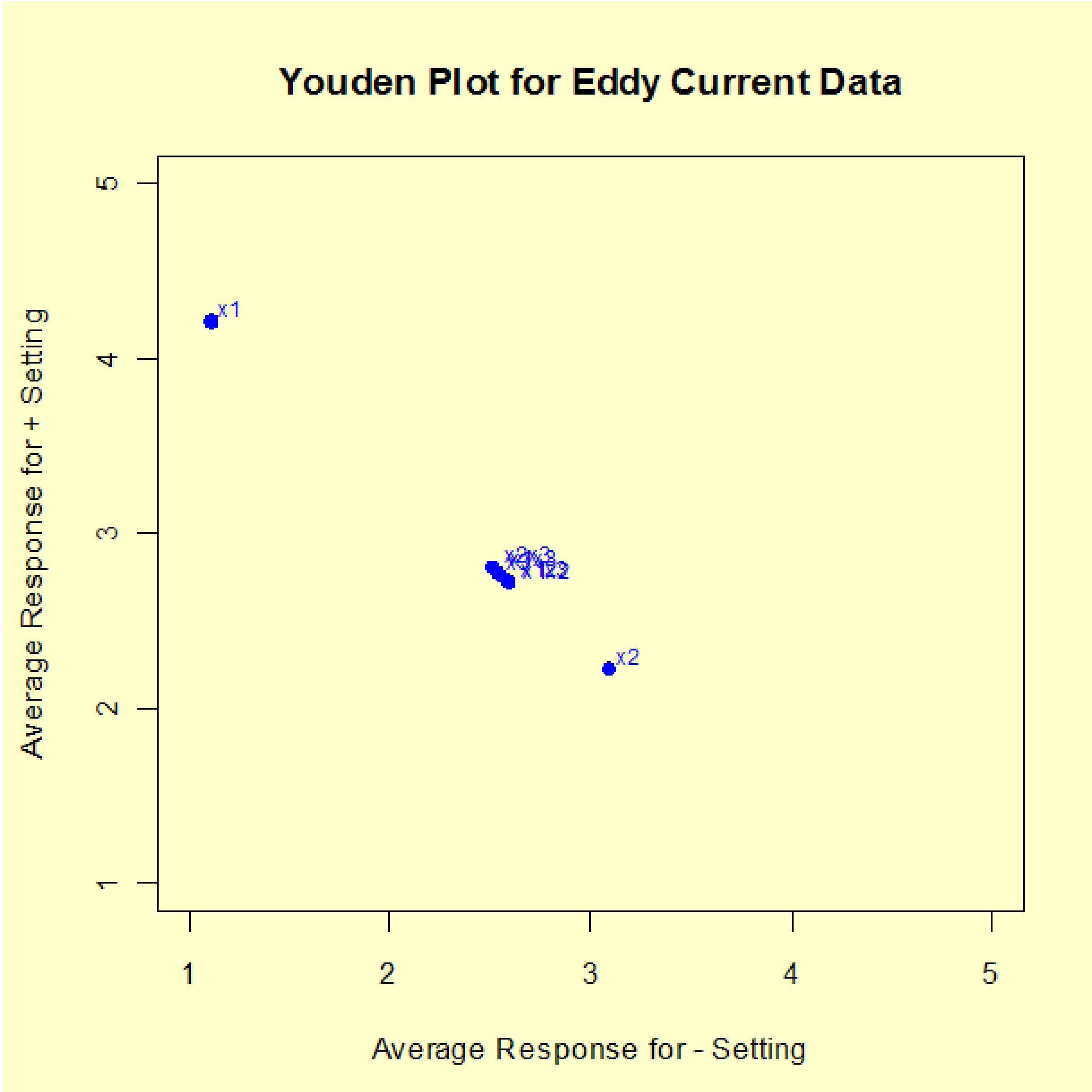
For the example at hand, the probability plot clearly shows two factors (X1 and X2) displaced off the line. All of the remaining five parameters are behaving like random drawings from a normal distribution centered at zero, and so are deemed to be statistically non-significant. In conclusion, this rule keeps two factors: X1 (1.55125) and X2 (-0.43375).

Averages: Youden Plot

A Youden plot can be used in the following way. Keep a factor as "important" if it is displaced away from the central-tendancy "bunch" in a Youden plot of high and low averages. By definition, a factor is important when its average response for the low (-1) setting is significantly different from its average response for the high (+1) setting. (Note that effects are twice the parameter estimates.) Conversely, if the low and high averages are about the same, then what difference does it make which setting to use and so why would such a factor be considered important? This fact in combination with the intrinsic benefits of the Youden plot for comparing pairs of items leads to the technique of generating a Youden plot of the low and high averages.

Youden Plot of Effect Estimates

The following is the Youden plot of the effect estimatess for the eddy current data.



For the example at hand, the Youden plot clearly shows a cluster of points near the grand average (2.65875) with two displaced points above (factor 1) and below (factor 2). Based on the Youden plot, we conclude to keep two factors: X1 (1.55125) and X2 (-0.43375).

Residual Standard Deviation: Engineering Significance

This criterion is defined as

Residual Standard Deviation > Cutoff

That is, declare a factor as "important" if the cumulative model that includes the factor (and all larger factors) has a residual standard deviation smaller than an a priori engineering-specified minimum residual standard deviation.

This criterion is different from the others in that it is model focused. In practice, this criterion states that starting with the largest parameter, we cumulatively keep adding terms to the model and monitor how the residual standard deviation for each progressively more complicated model becomes smaller. At some point, the cumulative model will become complicated enough and comprehensive enough that the resulting residual standard deviation will drop below the pre-specified engineering cutoff for the residual standard deviation. At that point, we stop adding terms and declare all of the model-included terms to be "important" and everything not in the model to be "unimportant".

This approach implies that the engineer has considered what a minimum residual standard deviation should be. In effect, this relates to what the engineer can tolerate for the magnitude of the typical residual (the difference between the raw data and the predicted value from the model). In other words, how good does the engineer want the prediction equation to be. Unfortunately, this engineering specification has not always been formulated and so this criterion can become moot.

In the absence of a prior specified cutoff, a good rough rule for the minimum engineering residual standard deviation is to keep adding terms until the residual standard deviation just dips below, say, 5 % of the current production average. For the eddy current data, let's say that the average detector has a sensitivity of 2.5 ohms. Then this would suggest that we would keep adding terms to the model until the residual standard deviation falls below 5 % of 2.5 ohms = 0.125 ohms.

```
                                                               Residual
Model                                                          Std. Dev.
-----------------------------------------------------          ---------
Mean + X1                                                        0.57272
Mean + X1 + X2                                                   0.30429
Mean + X1 + X2 + X2*X3                                           0.26737
Mean + X1 + X2 + X2*X3 + X1*X3                                   0.23341
Mean + X1 + X2 + X2*X3 + X1*X3 + X3                              0.19121
Mean + X1 + X2 + X2*X3 + X1*X3 + X3 + X1*X2*X3                   0.18031
Mean + X1 + X2 + X2*X3 + X1*X3 + X3 + X1*X2*X3 + X1*X2                NA
```

Based on the minimum residual standard deviation criteria, and we would include **all** terms in order to drive the residual standard deviation below 0.125. Again, the 5 % rule is a rough-and-ready rule that has no basis in engineering or statistics, but is simply a "numerics". Ideally, the engineer has a better cutoff for the residual standard deviation that is based on how well he/she wants the equation to peform in practice. If such a number were available, then for this criterion and data set we would select something less than the entire collection of terms.

Residual Standard Deviation: Statistical Significance

This criterion is defined as

Residual Standard Deviation > σ

where σ is the standard deviation of an observation under replicated conditions.

That is, declare a term as "important" until the cumulative model that includes the term has a residual standard deviation smaller than σ. In essence, we are allowing that we cannot demand a model fit any better than what we would obtain if we had replicated data; that is, we cannot demand that the residual standard deviation from any fitted model be any smaller than the (theoretical or actual) replication standard deviation. We can drive the fitted standard deviation down (by adding terms) until it achieves a value close to σ, but to attempt to drive it down further means that we are, in effect, trying to fit noise.

In practice, this criterion may be difficult to apply because

1. the engineer may not know σ;
2. the experiment might not have replication, and so a model-free estimate of σ is not obtainable.

For the current case study:

1. the engineer did **not** know σ;

2. the design (a 2^3 full factorial) did **not** have replication. The most common way of having replication in such designs is to have replicated center points at the center of the cube ((X1,X2,X3) = (0,0,0)).

Thus for this current case, this criteria could **not** be used to yield a subset of "important" factors.

Conclusions

In summary, the seven criteria for specifying "important" factors yielded the following for the eddy current data:

1. Parameters, Engineering Significance: X1, X2
2. Parameters, Numerically Significant: X1, X2
3. Parameters, Statistically Significant: X1, X2, X2*X3
4. Parameters, Probability Plots: X1, X2
5. Effects, Youden Plot: X1, X2
6. Residual SD, Engineering Significance: all 7 terms
7. Residual SD, Statistical Significance: not applicable

Such conflicting results are common. Arguably, the three most important criteria (listed in order of most important) are:

4. Parameters, Probability Plots: X1, X2
1. Parameters, Engineering Significance: X1, X2
3. Residual SD, Engineering Significance: all 7 terms

Scanning all of the above, we thus declare the following consensus for the eddy current data:

1. Important Factors: X1 and X2
2. Parsimonious Prediction Equation:

$$\hat{Y} = 2.65875 + 1.55125 \cdot X_1 - 0.43375 \cdot X_2$$

(with a residual standard deviation of 0.30429 ohms)

Note that this is the initial model selection. We still need to perform model validation with a residual analysis.

1.3.6. Probability Distributions

Probability Distributions

Probability distributions are a fundamental concept in statistics. They are used both on a theoretical level and a practical level.

Some practical uses of probability distributions are:

- To calculate confidence intervals for parameters and to calculate critical regions for hypothesis tests.
- For univariate data, it is often useful to determine a reasonable distributional model for the data.
- Statistical intervals and hypothesis tests are often based on specific distributional assumptions. Before computing an interval or test based on a distributional assumption, we need to verify that the assumption is justified for the given data set. In this case, the distribution does not need to be the best-fitting distribution for the data, but an adequate enough model so that the statistical technique yields valid conclusions.
- Simulation studies with random numbers generated from using a specific probability distribution are often needed.

Table of Contents

1.3.6.1. What is a Probability Distribution

Discrete Distributions

The mathematical definition of a discrete probability function, p(x), is a function that satisfies the following properties.

1. The probability that x can take a specific value is p(x). That is

   $$P[X = x] = p(x) = p_x$$

2. p(x) is non-negative for all real x.

3. The sum of p(x) over all possible values of x is 1, that is

   $$\sum_j p_j = 1$$

   where ***j*** represents all possible values that x can have and p_j is the probability at x_j.

   One consequence of properties 2 and 3 is that 0 <= p(x) <= 1.

What does this actually mean? A discrete probability function is a function that can take a discrete number of values (not necessarily finite). This is most often the non-negative integers or some subset of the non-negative integers. There is no mathematical restriction that discrete probability functions only be defined at integers, but in practice this is usually what makes sense. For example, if you toss a coin 6 times, you can get 2 heads or 3 heads but not 2 1/2 heads. Each of the discrete values has a certain probability of occurrence that is between zero and one. That is, a discrete function that allows negative values or values greater than one is not a probability function. The condition that the probabilities sum to one means that at least one of the values has to occur.

Continuous Distributions

The mathematical definition of a continuous probability function, f(x), is a function that satisfies the following properties.

1. The probability that x is between two points a and b is

$$p[a \leq x \leq b] = \int_a^b f(x)dx$$

2. It is non-negative for all real x.

3. The integral of the probability function is one, that is

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

What does this actually mean? Since continuous probability functions are defined for an infinite number of points over a continuous interval, the probability at a single point is always zero. Probabilities are measured over intervals, not single points. That is, the area under the curve between two distinct points defines the probability for that interval. This means that the height of the probability function can in fact be greater than one. The property that the integral must equal one is equivalent to the property for discrete distributions that the sum of all the probabilities must equal one.

Probability Mass Functions Versus Probability Density Functions

Discrete probability functions are referred to as probability mass functions and continuous probability functions are referred to as probability density functions. The term probability functions covers both discrete and continuous distributions. When we are referring to probability functions in generic terms, we may use the term probability density functions to mean both discrete and continuous probability functions.

1.3.6.2. Related Distributions

Probability distributions are typically defined in terms of the probability density function. However, there are a number of probability functions used in applications.

Probability Density Function

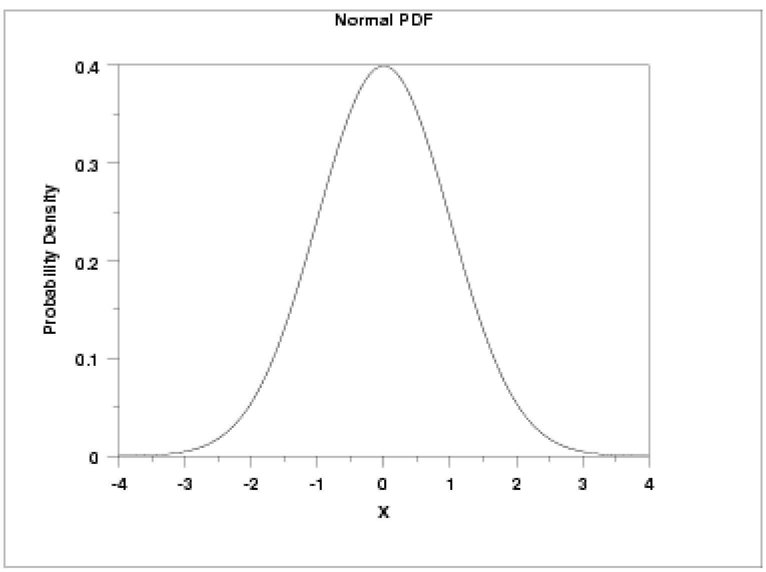
For a continuous function, the probability density function (pdf) is the probability that the variate has the value x. Since for continuous distributions the probability at a single point is zero, this is often expressed in terms of an integral between two points.

$$\int_a^b f(x)dx = Pr[a \le X \le b]$$

For a discrete distribution, the pdf is the probability that the variate takes the value x.

$$f(x) = Pr[X = x]$$

The following is the plot of the normal probability density function.



Cumulative Distribution Function

The cumulative distribution function (cdf) is the probability that the variable takes a value less than or equal to x. That is

$$F(x) = Pr[X \le x] = \alpha$$

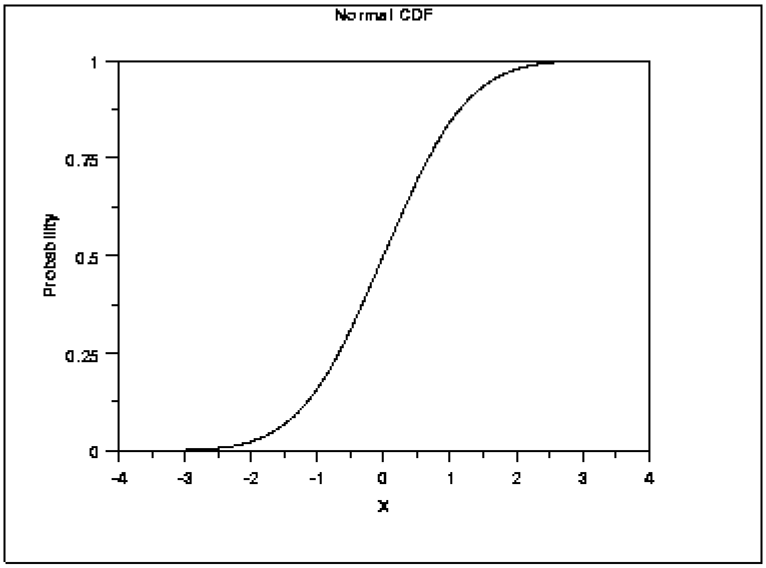
For a continuous distribution, this can be expressed mathematically as

$$F(x) = \int_{-\infty}^{x} f(\mu) d\mu$$

For a discrete distribution, the cdf can be expressed as

$$F(x) = \sum_{i=0}^{x} f(i)$$

The following is the plot of the normal cumulative distribution function.



The horizontal axis is the allowable domain for the given probability function. Since the vertical axis is a probability, it must fall between zero and one. It increases from zero to one as we go from left to right on the horizontal axis.

Percent Point Function

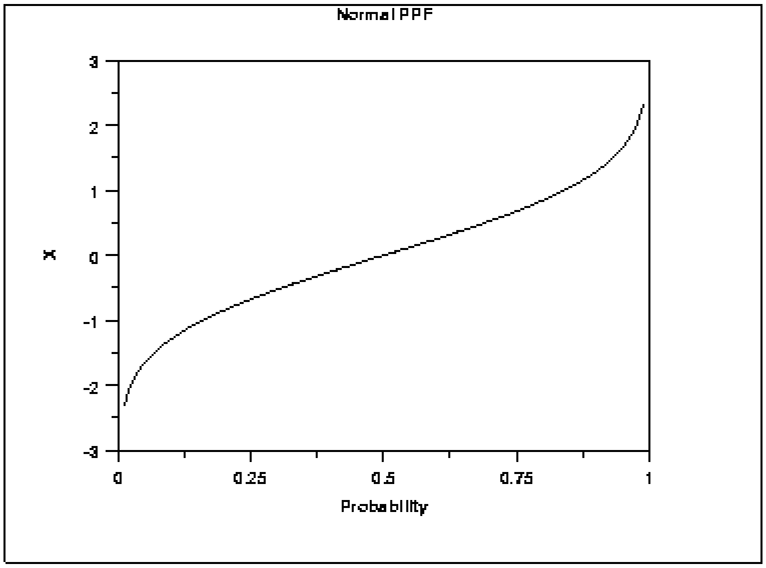
The percent point function (ppf) is the inverse of the cumulative distribution function. For this reason, the percent point function is also commonly referred to as the inverse distribution function. That is, for a distribution function we calculate the probability that the variable is less than or equal to x for a given x. For the percent point function, we start with the probability and compute the corresponding x for the cumulative distribution. Mathematically, this can be expressed as

$$Pr[X \leq G(\alpha)] = \alpha$$

or alternatively

$$x = G(\alpha) = G(F(x))$$

The following is the plot of the normal percent point function.



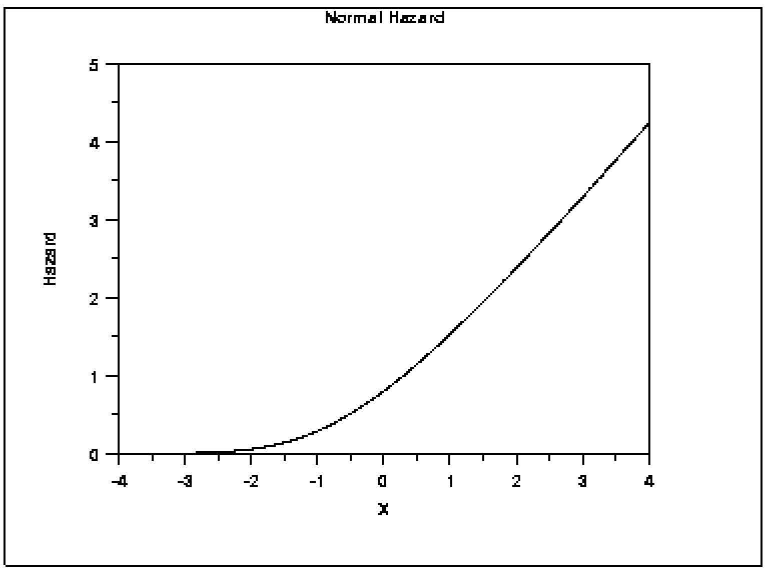
Since the horizontal axis is a probability, it goes from zero to one. The vertical axis goes from the smallest to the largest value of the cumulative distribution function.

Hazard Function

The hazard function is the ratio of the probability density function to the survival function, *S*(x).

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}$$

The following is the plot of the normal distribution hazard function.



Hazard plots are most commonly used in reliability applications. Note that Johnson, Kotz, and Balakrishnan refer to this as the conditional failure density function rather than the hazard function.

Cumulative Hazard

The cumulative hazard function is the integral of the hazard function.

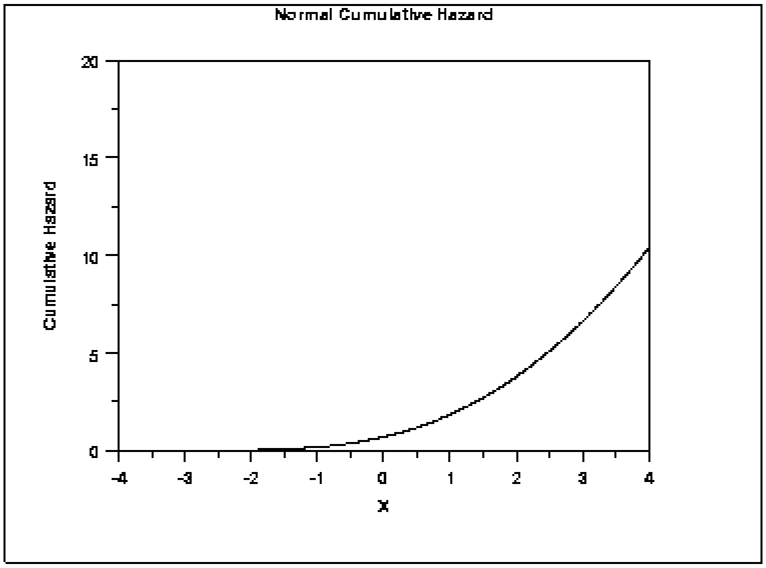
Function

$$H(x) = \int_{-\infty}^{x} h(\mu) d\mu$$

This can alternatively be expressed as

$$H(x) = -\ln\left(1 - F(x)\right)$$

The following is the plot of the normal cumulative hazard function.



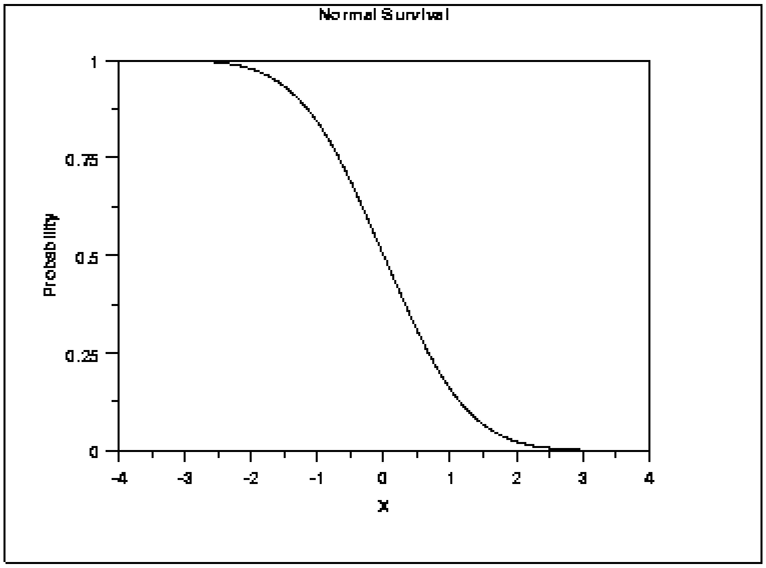
Cumulative hazard plots are most commonly used in reliability applications. Note that Johnson, Kotz, and Balakrishnan refer to this as the hazard function rather than the cumulative hazard function.

Survival Function

Survival functions are most often used in reliability and related fields. The survival function is the probability that the variate takes a value greater than x.

$$S(x) = Pr[X > x] = 1 - F(x)$$

The following is the plot of the normal distribution survival function.



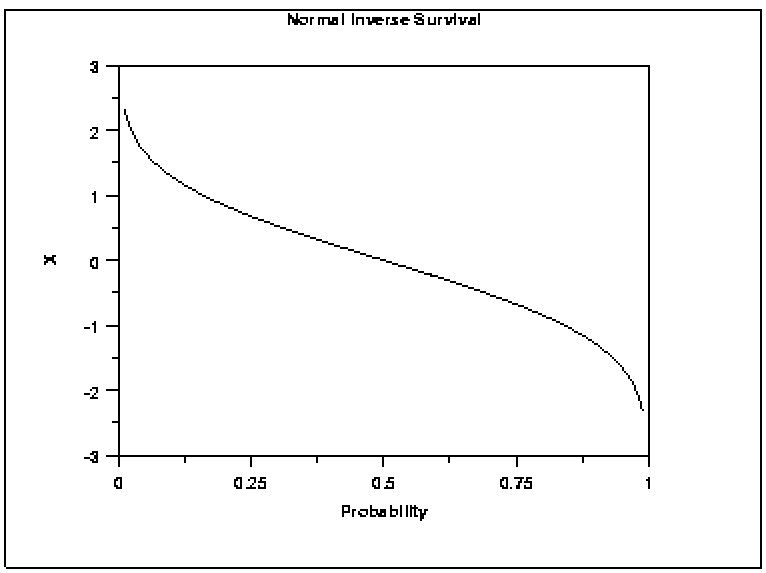
For a survival function, the y value on the graph starts at 1 and monotonically decreases to zero. The survival function should be compared to the cumulative distribution function.

Inverse Survival Function

Just as the percent point function is the inverse of the cumulative distribution function, the survival function also has an inverse function. The inverse survival function can be defined in terms of the percent point function.

$$Z(\alpha) = G(1 - \alpha)$$

The following is the plot of the normal distribution inverse survival function.



As with the percent point function, the horizontal axis is a probability. Therefore the horizontal axis goes from 0 to 1 regardless of the particular distribution. The appearance is similar to the percent point function. However, instead of going from the smallest to the largest value on the vertical axis, it goes from the largest to the smallest value.

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.6. Probability Distributions

1.3.6.3. Families of Distributions

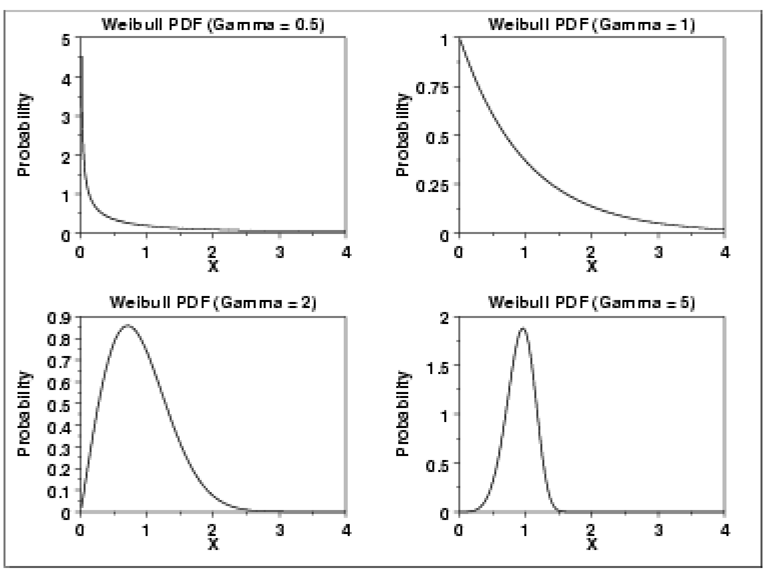
Shape Parameters

Many probability distributions are not a single distribution, but are in fact a family of distributions. This is due to the distribution having one or more shape parameters.

Shape parameters allow a distribution to take on a variety of shapes, depending on the value of the shape parameter. These distributions are particularly useful in modeling applications since they are flexible enough to model a variety of data sets.

Example: Weibull Distribution

The Weibull distribution is an example of a distribution that has a shape parameter. The following graph plots the Weibull pdf with the following values for the shape parameter: 0.5, 1.0, 2.0, and 5.0.



The shapes above include an exponential distribution, a right-skewed distribution, and a relatively symmetric distribution.

The Weibull distribution has a relatively simple distributional form. However, the shape parameter allows the Weibull to assume a wide variety of shapes. This combination of simplicity and flexibility in the shape of the Weibull distribution has made it an effective distributional model in reliability applications. This ability to model a wide variety of distributional shapes using a relatively simple distributional form is possible with many other distributional families as well.

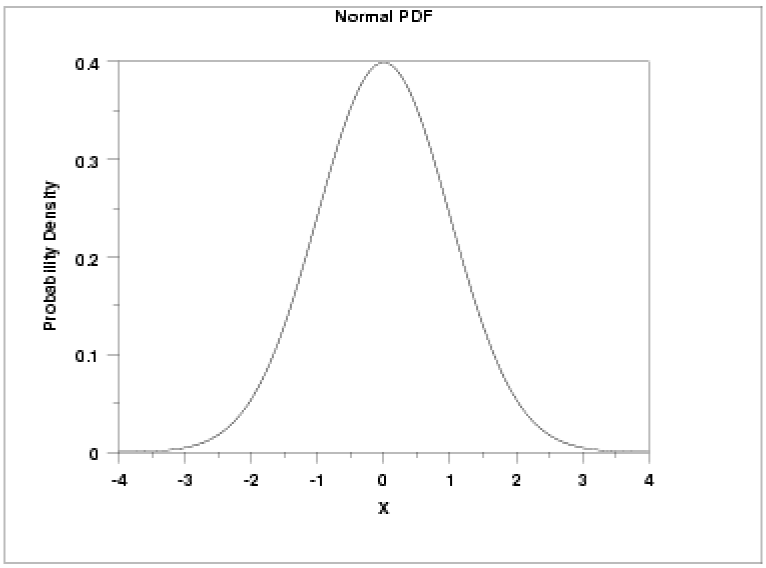
PPCC Plots The PPCC plot is an effective graphical tool for selecting the member of a distributional family with a single shape parameter that best fits a given set of data.

1.3.6.4. Location and Scale Parameters

Normal PDF

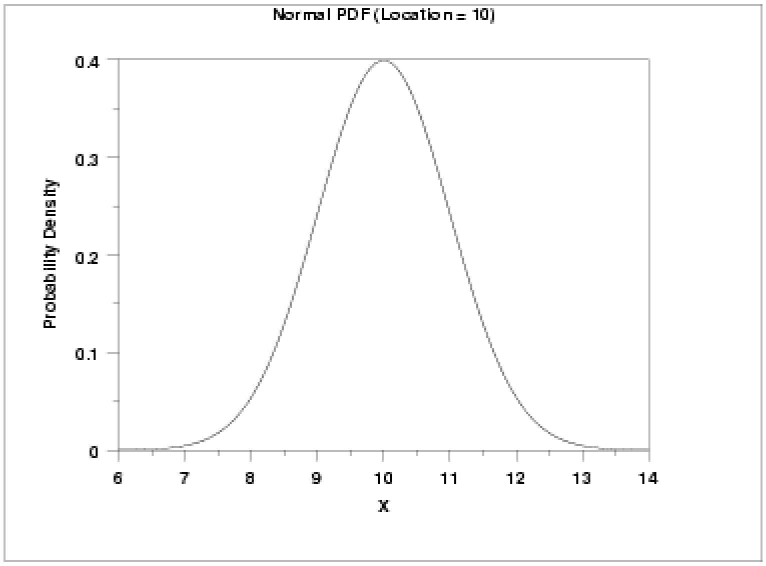
A probability distribution is characterized by location and scale parameters. Location and scale parameters are typically used in modeling applications.

For example, the following graph is the probability density function for the standard normal distribution, which has the location parameter equal to zero and scale parameter equal to one.



Location Parameter

The next plot shows the probability density function for a normal distribution with a location parameter of 10 and a scale parameter of 1.

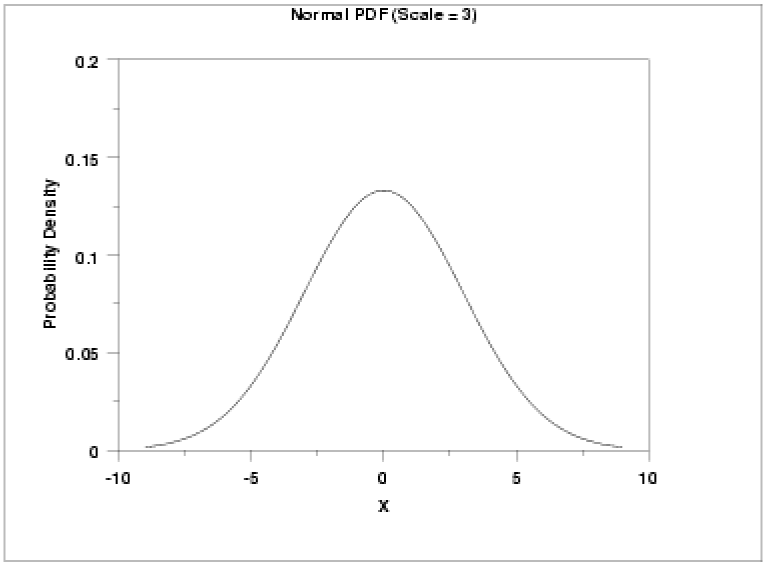


The effect of the location parameter is to translate the graph, relative to the standard normal distribution, 10 units to the right on the horizontal axis. A location parameter of -10 would have shifted the graph 10 units to the left on the horizontal axis.

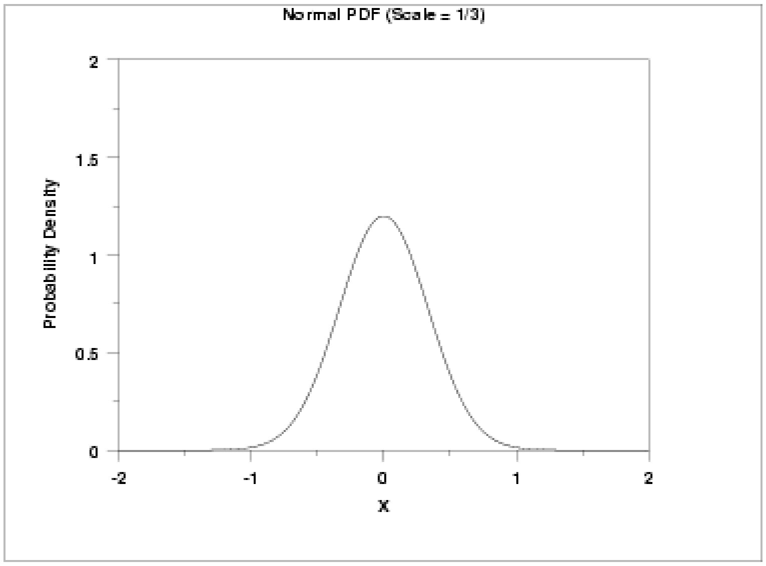
That is, a location parameter simply shifts the graph left or right on the horizontal axis.

Scale Parameter

The next plot has a scale parameter of 3 (and a location parameter of zero). The effect of the scale parameter is to stretch out the graph. The maximum y value is approximately 0.13 as opposed 0.4 in the previous graphs. The y value, i.e., the vertical axis value, approaches zero at about (+/-) 9 as opposed to (+/-) 3 with the first graph.



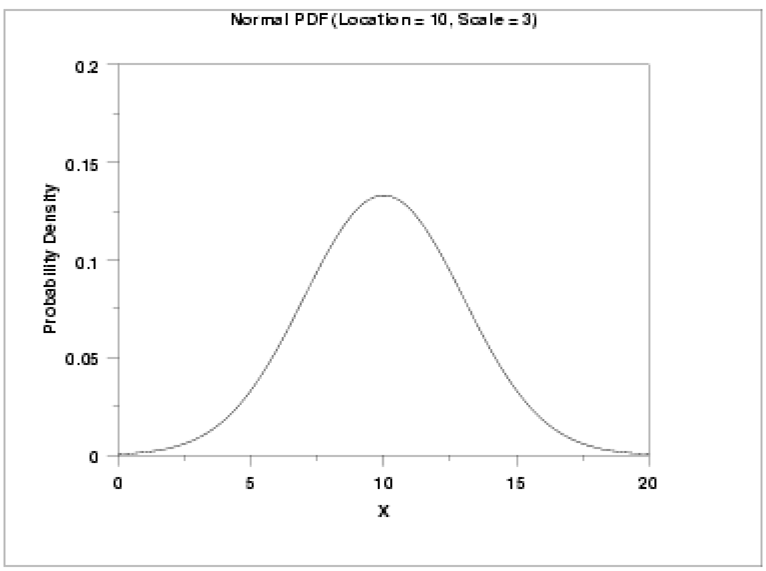
In contrast, the next graph has a scale parameter of 1/3 (=0.333). The effect of this scale parameter is to squeeze the pdf. That is, the maximum y value is approximately 1.2 as opposed to 0.4 and the y value is near zero at (+/-) 1 as opposed to (+/-) 3.



The effect of a scale parameter greater than one is to stretch the pdf. The greater the magnitude, the greater the stretching. The effect of a scale parameter less than one is to compress the pdf. The compressing approaches a spike as the scale parameter goes to zero. A scale parameter of 1 leaves the pdf unchanged (if the scale parameter is 1 to begin with) and non-positive scale parameters are not allowed.

Location and Scale Together

The following graph shows the effect of both a location and a scale parameter. The plot has been shifted right 10 units and stretched by a factor of 3.



Standard Form

The standard form of any distribution is the form that has location parameter zero and scale parameter one.

It is common in statistical software packages to only compute the standard form of the distribution. There are formulas for converting from the standard form to the form with other location and scale parameters. These formulas are independent of the particular probability distribution.

Formulas for Location and Scale Based on the Standard Form

The following are the formulas for computing various probability functions based on the standard form of the distribution. The parameter *a* refers to the location parameter and the parameter *b* refers to the scale parameter. Shape parameters are not included.

Cumulative Distribution Function	$F(x;a,b) = F((x-a)/b;0,1)$
Probability Density Function	$f(x;a,b) = (1/b)f((x-a)/b;0,1)$
Percent Point Function	$G(\alpha;a,b) = a + bG(\alpha;0,1)$
Hazard Function	$h(x;a,b) = (1/b)h((x-a)/b;0,1)$
Cumulative Hazard Function	$H(x;a,b) = H((x-a)/b;0,1)$
Survival Function	$S(x;a,b) = S((x-a)/b;0,1)$
Inverse Survival Function	$Z(\alpha;a,b) = a + bZ(\alpha;0,1)$
Random Numbers	$Y(a,b) = a + bY(0,1)$

Relationship to Mean and Standard Deviation

For the normal distribution, the location and scale parameters correspond to the mean and standard deviation, respectively. However, this is not necessarily true for other distributions. In fact, it is not true for most distributions.

1.3.6.5. Estimating the Parameters of a Distribution

Model a univariate data set with a probability distribution

One common application of probability distributions is modeling univariate data with a specific probability distribution. This involves the following two steps:

1. Determination of the "best-fitting" distribution.
2. Estimation of the parameters (shape, location, and scale parameters) for that distribution.

Various Methods

There are various methods, both numerical and graphical, for estimating the parameters of a probability distribution.

1. Method of moments
2. Maximum likelihood
3. Least squares
4. PPCC and probability plots

1.3.6.5.1. Method of Moments

Method of Moments

The method of moments equates sample moments to parameter estimates. When moment methods are available, they have the advantage of simplicity. The disadvantage is that they are often not available and they do not have the desirable optimality properties of maximum likelihood and least squares estimators.

The primary use of moment estimates is as starting values for the more precise maximum likelihood and least squares estimates.

Software

Most general purpose statistical software does not include explicit method of moments parameter estimation commands. However, when utilized, the method of moment formulas tend to be straightforward and can be easily implemented in most statistical software programs.

1.3.6.5.2. Maximum Likelihood

Maximum Likelihood

Maximum likelihood estimation begins with the mathematical expression known as a likelihood function of the sample data. Loosely speaking, the likelihood of a set of data is the probability of obtaining that particular set of data given the chosen probability model. This expression contains the unknown parameters. Those values of the parameter that maximize the sample likelihood are known as the maximum likelihood estimates.

The reliability chapter contains some examples of the likelihood functions for a few of the commonly used distributions in reliability analysis.

Advantages

The advantages of this method are:

- Maximum likelihood provides a consistent approach to parameter estimation problems. This means that maximum likelihood estimates can be developed for a large variety of estimation situations. For example, they can be applied in reliability analysis to censored data under various censoring models.

- Maximum likelihood methods have desirable mathematical and optimality properties. Specifically,
    1. They become minimum variance unbiased estimators as the sample size increases. By unbiased, we mean that if we take (a very large number of) random samples with replacement from a population, the average value of the parameter estimates will be theoretically exactly equal to the population value. By minimum variance, we mean that the estimator has the smallest variance, and thus the narrowest confidence interval, of all estimators of that type.
    2. They have approximate normal distributions and approximate sample variances that can be used to generate confidence bounds and hypothesis tests for the parameters.

- Several popular statistical software packages provide excellent algorithms for maximum likelihood estimates for many of the commonly used distributions. This helps mitigate the computational complexity of maximum likelihood estimation.

Disadvantages

The disadvantages of this method are:

- The likelihood equations need to be specifically worked out for a given distribution and estimation problem. The mathematics is often non-trivial, particularly if confidence intervals for the parameters are desired.
- The numerical estimation is usually non-trivial. Except for a few cases where the maximum likelihood formulas are in fact simple, it is generally best to rely on high quality statistical software to obtain maximum likelihood estimates. Fortunately, high quality maximum likelihood software is becoming increasingly common.
- Maximum likelihood estimates can be heavily biased for small samples. The optimality properties may not apply for small samples.
- Maximum likelihood can be sensitive to the choice of starting values.

Software

Most general purpose statistical software programs support maximum likelihood estimation (MLE) in some form. MLE estimation can be supported in two ways.

1. A software program may provide a generic function minimization (or equivalently, maximization) capability. This is also referred to as function optimization. Maximum likelihood estimation is essentially a function optimization problem.

   This type of capability is particularly common in mathematical software programs.

2. A software program may provide MLE computations for a specific problem. For example, it may generate ML estimates for the parameters of a Weibull distribution.

   Statistical software programs will often provide ML estimates for many specific problems even when they do not support general function optimization.

The advantage of function minimization software is that it can be applied to many different MLE problems. The drawback is that you have to specify the maximum likelihood equations to the software. As the functions can be non-trivial, there is potential for error in entering the equations.

The advantage of the specific MLE procedures is that greater efficiency and better numerical stability can often be obtained by taking advantage of the properties of the specific estimation problem. The specific methods often return explicit confidence intervals. In addition, you do not have to know or specify the likelihood equations to the software. The disadvantage is that each MLE problem must be specifically coded.

1.3.6.5.3. Least Squares

Least Squares

Non-linear least squares provides an alternative to maximum likelihood.

Advantages

The advantages of this method are:

- Non-linear least squares software may be available in many statistical software packages that do not support maximum likelihood estimates.
- It can be applied more generally than maximum likelihood. That is, if your software provides non-linear fitting and it has the ability to specify the probability function you are interested in, then you can generate least squares estimates for that distribution. This will allow you to obtain reasonable estimates for distributions even if the software does not provide maximum likelihood estimates.

Disadvantages

The disadvantages of this method are:

- It is not readily applicable to censored data.
- It is generally considered to have less desirable optimality properties than maximum likelihood.
- It can be quite sensitive to the choice of starting values.

Software

Non-linear least squares fitting is available in many general purpose statistical software programs.

1.3.6.5.4. PPCC and Probability Plots

PPCC and Probability Plots

The PPCC plot can be used to estimate the shape parameter of a distribution with a single shape parameter. After finding the best value of the shape parameter, the probability plot can be used to estimate the location and scale parameters of a probability distribution.

Advantages

The advantages of this method are:

- It is based on two well-understood concepts.
  1. The linearity (i.e., straightness) of the probability plot is a good measure of the adequacy of the distributional fit.
  2. The correlation coefficient between the points on the probability plot is a good measure of the linearity of the probability plot.
- It is an easy technique to implement for a wide variety of distributions with a single shape parameter. The basic requirement is to be able to compute the percent point function, which is needed in the computation of both the probability plot and the PPCC plot.
- The PPCC plot provides insight into the sensitivity of the shape parameter. That is, if the PPCC plot is relatively flat in the neighborhood of the optimal value of the shape parameter, this is a strong indication that the fitted model will not be sensitive to small deviations, or even large deviations in some cases, in the value of the shape parameter.
- The maximum correlation value provides a method for comparing across distributions as well as identifying the best value of the shape parameter for a given distribution. For example, we could use the PPCC and probability fits for the Weibull, lognormal, and possibly several other distributions. Comparing the maximum correlation coefficient achieved for each distribution can help in selecting which is the best distribution to use.

Disadvantages The disadvantages of this method are:

- It is limited to distributions with a single shape parameter.
- PPCC plots are not widely available in statistical software packages other than Dataplot (Dataplot provides PPCC plots for 40+ distributions). Probability plots are generally available. However, many statistical software packages only provide them for a limited number of distributions.
- Significance levels for the correlation coefficient (i.e., if the maximum correlation value is above a given value, then the distribution provides an adequate fit for the data with a given confidence level) have only been worked out for a limited number of distributions.

Other Graphical Methods For reliability applications, the hazard plot and the Weibull plot are alternative graphical methods that are commonly used to estimate parameters.

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.6. Probability Distributions

1.3.6.6. Gallery of Distributions

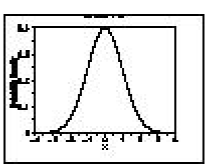
Gallery of Common Distributions

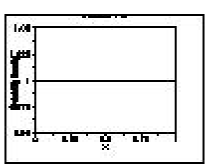
Detailed information on a few of the most common distributions is available below. There are a large number of distributions used in statistical applications. It is beyond the scope of this Handbook to discuss more than a few of these. Two excellent sources for additional detailed information on a large array of distributions are Johnson, Kotz, and Balakrishnan and Evans, Hastings, and Peacock. Equations for the probability functions are given for the standard form of the distribution. Formulas exist for defining the functions with location and scale parameters in terms of the standard form of the distribution.

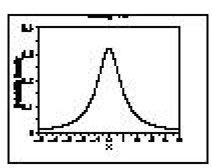
The sections on parameter estimation are restricted to the method of moments and maximum likelihood. This is because the least squares and PPCC and probability plot estimation procedures are generic. The maximum likelihood equations are not listed if they involve solving simultaneous equations. This is because these methods require sophisticated computer software to solve. Except where the maximum likelihood estimates are trivial, you should depend on a statistical software program to compute them. References are given for those who are interested.

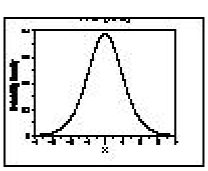
Be aware that different sources may give formulas that are different from those shown here. In some cases, these are simply mathematically equivalent formulations. In other cases, a different parameterization may be used.

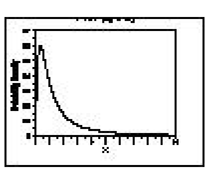
Continuous Distributions

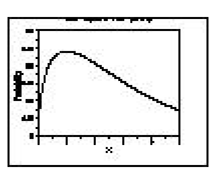

Normal Distribution


Uniform Distribution

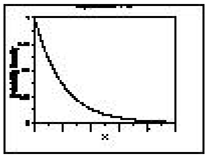

Cauchy Distribution

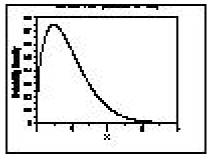


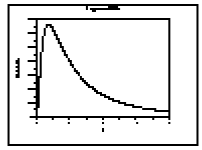


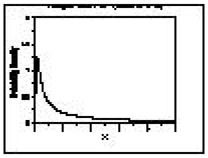


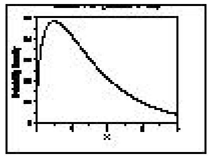
t Distribution	F Distribution	Chi-Square Distribution

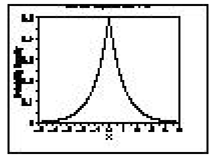

Exponential Distribution

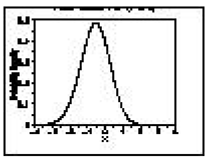

Weibull Distribution

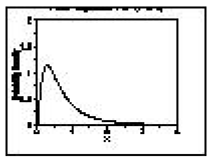

Lognormal Distribution

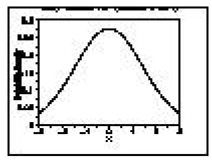

Birnbaum-Saunders (Fatigue Life) Distribution

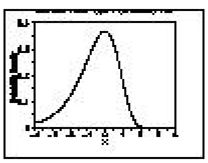

Gamma Distribution

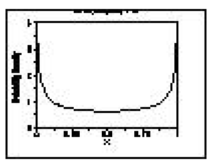

Double Exponential Distribution


Power Normal Distribution

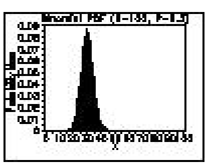

Power Lognormal Distribution

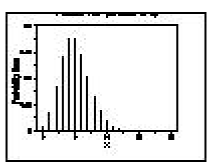

Tukey-Lambda Distribution


Extreme Value Type I Distribution


Beta Distribution

Discrete Distributions


Binomial Distribution


Poisson Distribution

1.3.6.6.1. Normal Distribution

Probability Density Function

The general formula for the probability density function of the normal distribution is

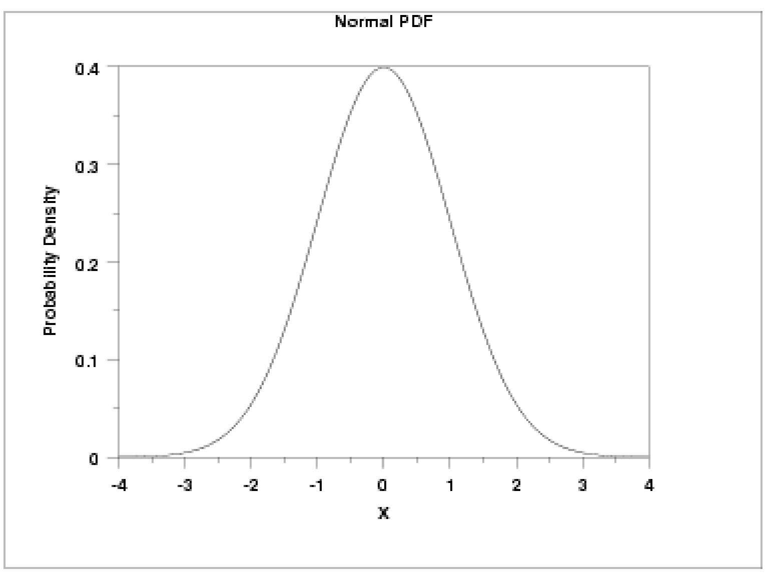
$$f(x) = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}$$

where μ is the location parameter and σ is the scale parameter. The case where μ = 0 and σ = 1 is called the **standard normal distribution**. The equation for the standard normal distribution is

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the standard normal probability density function.



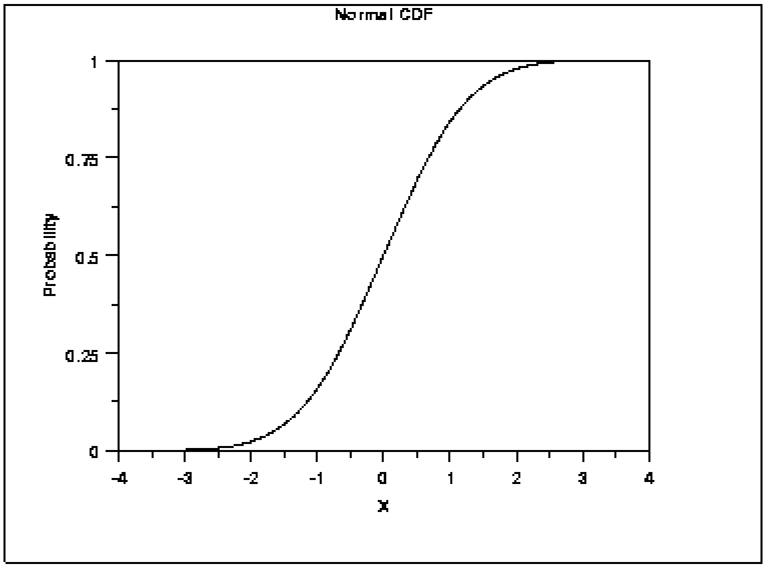
Cumulative

The formula for the cumulative distribution function of the

Distribution Function

normal distribution does not exist in a simple closed formula. It is computed numerically.

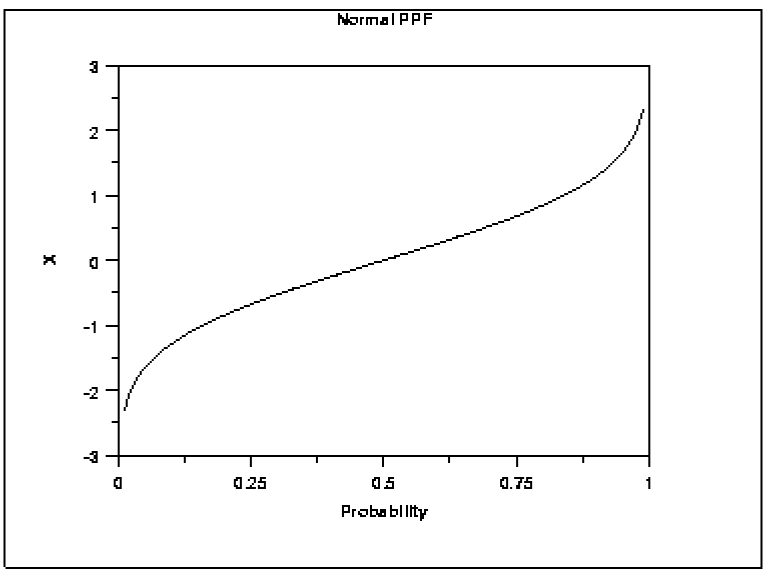
The following is the plot of the normal cumulative distribution function.



Percent Point Function

The formula for the percent point function of the normal distribution does not exist in a simple closed formula. It is computed numerically.

The following is the plot of the normal percent point function.



Hazard Function

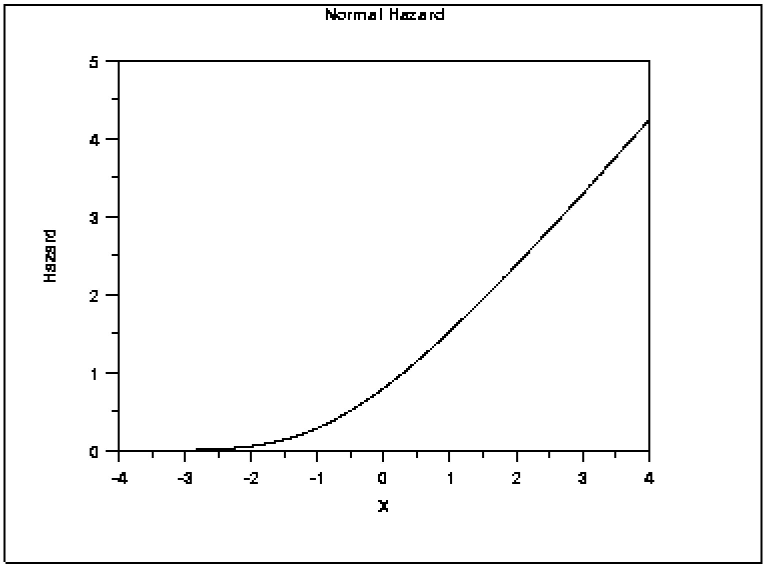
The formula for the hazard function of the normal distribution is

$$h(x) = \frac{\phi(x)}{\Phi(-x)}$$

where Φ is the cumulative distribution function of the ϕ

standard normal distribution and is the probability density function of the standard normal distribution.

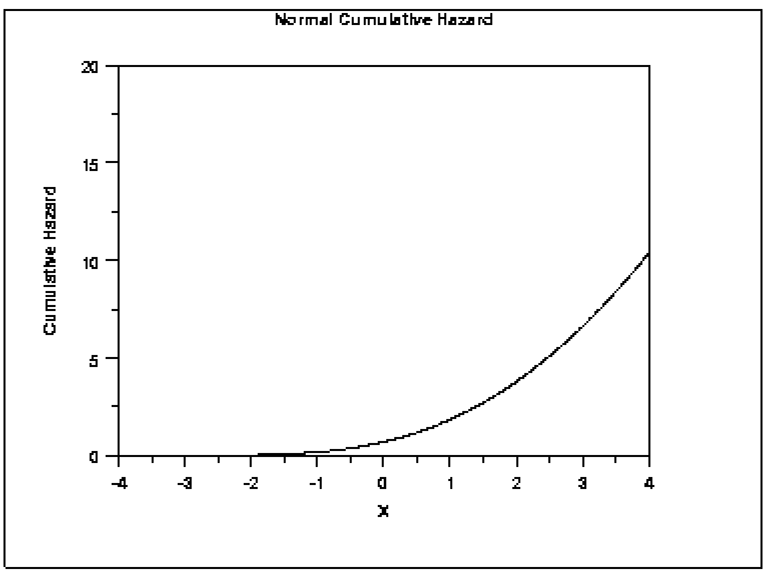
The following is the plot of the normal hazard function.



Cumulative Hazard Function

The normal cumulative hazard function can be computed from the normal cumulative distribution function.

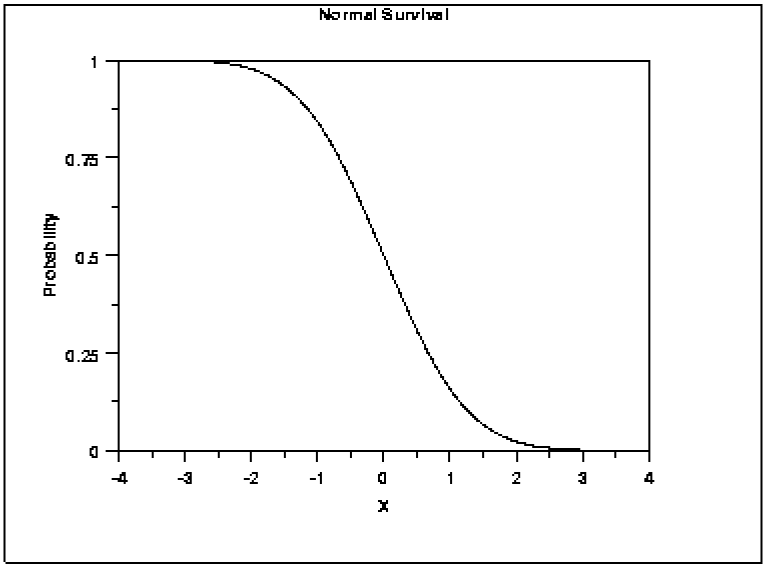
The following is the plot of the normal cumulative hazard function.



Survival Function

The normal survival function can be computed from the normal cumulative distribution function.

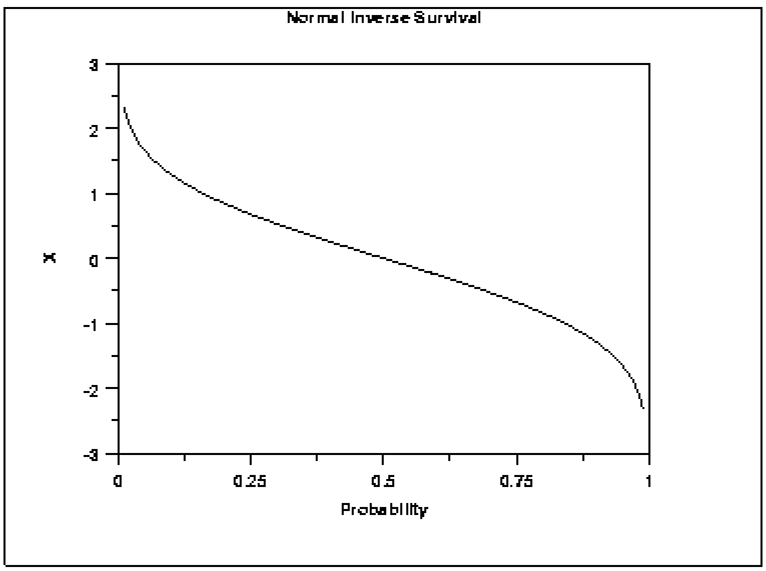
The following is the plot of the normal survival function.



Inverse Survival Function

The normal inverse survival function can be computed from the normal percent point function.

The following is the plot of the normal inverse survival function.



Common Statistics

Mean	The location parameter μ.
Median	The location parameter μ.
Mode	The location parameter μ.
Range	Infinity in both directions.
Standard Deviation	The scale parameter σ.
Coefficient of Variation	σ/μ
Skewness	0
Kurtosis	3

Parameter The location and scale parameters of the normal distribution

Estimation	can be estimated with the sample mean and sample standard deviation, respectively.
Comments	For both theoretical and practical reasons, the normal distribution is probably the most important distribution in statistics. For example, • Many classical statistical tests are based on the assumption that the data follow a normal distribution. This assumption should be tested before applying these tests. • In modeling applications, such as linear and non-linear regression, the error term is often assumed to follow a normal distribution with fixed location and scale. • The normal distribution is used to find significance levels in many hypothesis tests and confidence intervals.
Theroretical Justification - Central Limit Theorem	The normal distribution is widely used. Part of the appeal is that it is well behaved and mathematically tractable. However, the central limit theorem provides a theoretical basis for why it has wide applicability. The central limit theorem basically states that as the sample size (*N*) becomes large, the following occur: 1. The sampling distribution of the mean becomes approximately normal regardless of the distribution of the original variable. 2. The sampling distribution of the mean is centered at the population mean, μ, of the original variable. In addition, the standard deviation of the sampling distribution of the mean approaches $\sigma/\sqrt{N}$.
Software	Most general purpose statistical software programs support at least some of the probability functions for the normal distribution.

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.6. Probability Distributions
1.3.6.6. Gallery of Distributions

1.3.6.6.2. Uniform Distribution

Probability Density Function

The general formula for the probability density function of the uniform distribution is

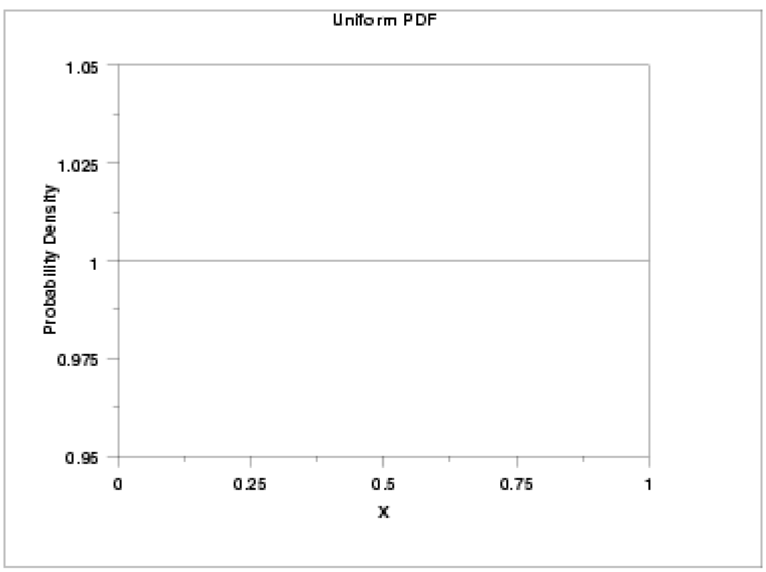
$$f(x) = \frac{1}{B-A} \quad \mbox{for } A \le x \le B$$

where A is the location parameter and (B - A) is the scale parameter. The case where A = 0 and B = 1 is called the **standard uniform distribution**. The equation for the standard uniform distribution is

$$f(x) = 1 \quad \mbox{for } 0 \le x \le 1$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the uniform probability density function.

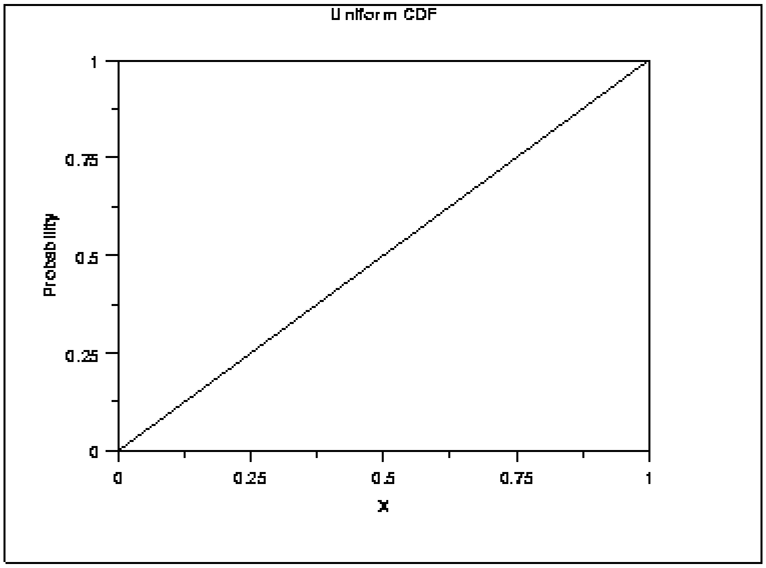


Cumulative Distribution Function

The formula for the cumulative distribution function of the uniform distribution is

$$F(x) = x \quad \mbox{for } 0 \le x \le 1$$

The following is the plot of the uniform cumulative distribution function.

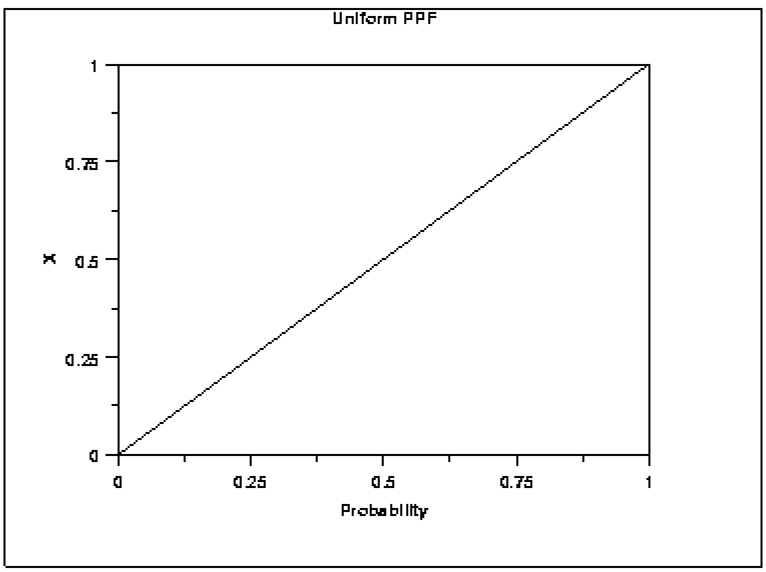


Percent Point Function

The formula for the percent point function of the uniform distribution is

$$G(p) = p \qquad \text{for } 0 \le p \le 1$$

The following is the plot of the uniform percent point function.

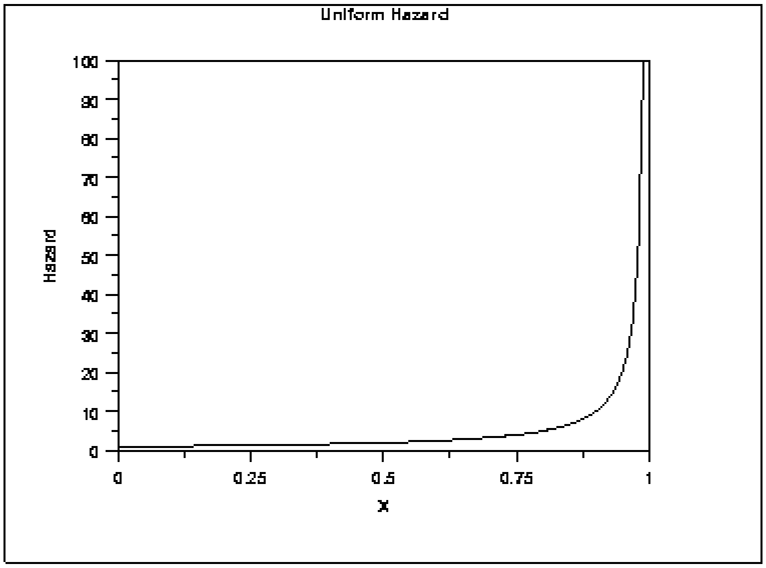


Hazard Function

The formula for the hazard function of the uniform distribution is

$$h(x) = \frac{1}{1-x} \qquad \text{for } 0 \le x < 1$$

The following is the plot of the uniform hazard function.

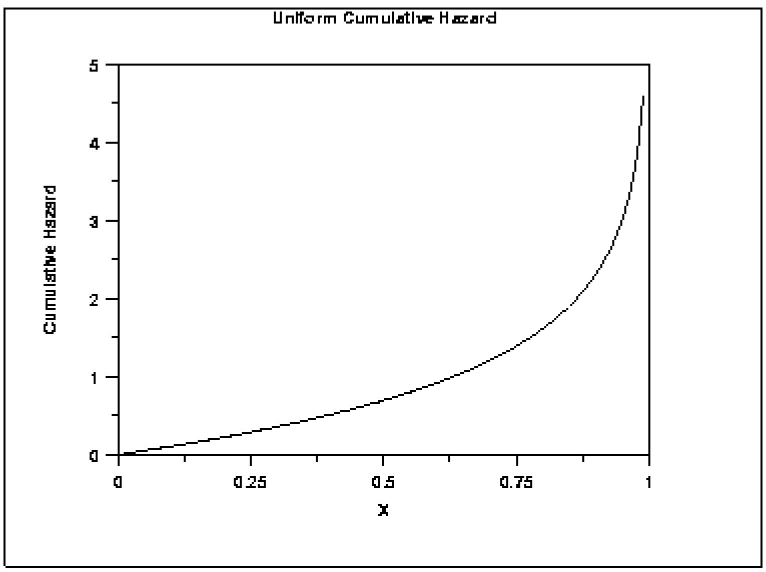


Cumulative Hazard Function

The formula for the cumulative hazard function of the uniform distribution is

$$H(x) = -ln(1 - x) \qquad \text{for } 0 \le x < 1$$

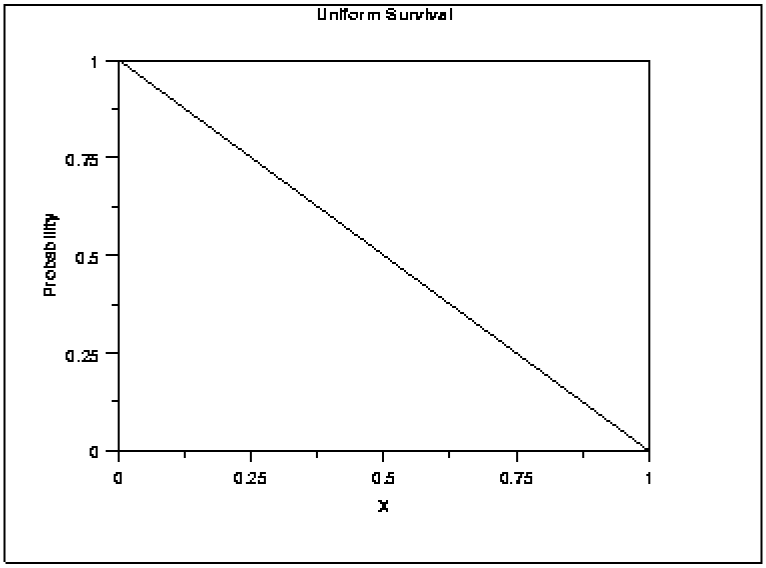
The following is the plot of the uniform cumulative hazard function.



Survival Function

The uniform survival function can be computed from the uniform cumulative distribution function.

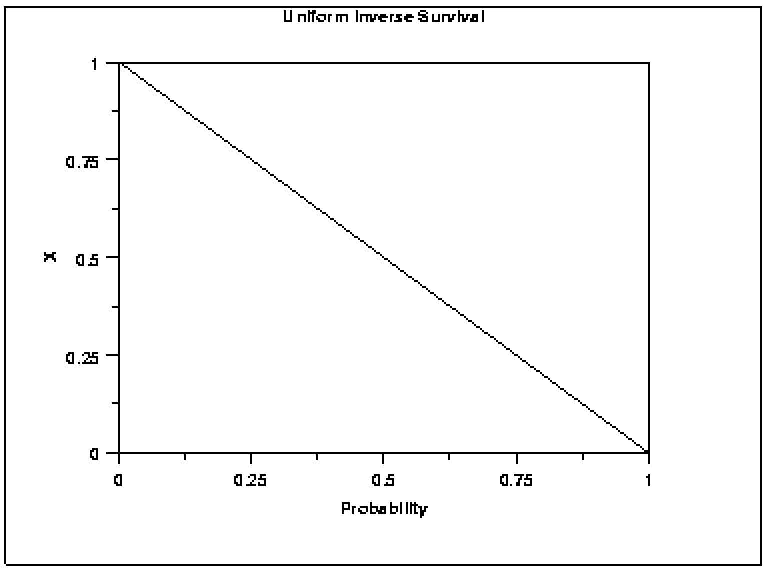
The following is the plot of the uniform survival function.



Inverse Survival Function

The uniform inverse survival function can be computed from the uniform percent point function.

The following is the plot of the uniform inverse survival function.



Common Statistics

Mean	(A + B)/2
Median	(A + B)/2
Range	B - A
Standard Deviation	$\sqrt{\frac{(B-A)^2}{12}}$
Coefficient of Variation	$\frac{(B-A)}{\sqrt{3}(B+A)}$
Skewness	0
Kurtosis	9/5

Parameter Estimation

The method of moments estimators for A and B are

$$\hat{A} = \bar{x} - \sqrt{3}s$$

$$\hat{B} = \bar{x} + \sqrt{3}s$$

The maximum likelihood estimators are usually given in terms of the parameters *a* and *h* where

A = *a* - *h*
B = *a* + *h*

The maximum likelihood estimators for *a* and *h* are

$$\hat{a} = \mathrm{midrange}(Y_1, Y_2, \ldots, Y_n)$$
$$\hat{h} = 0.5[\mathrm{range}(Y_1, Y_2, \ldots, Y_n)]$$

This gives the following maximum likelihood estimators for A and B

$$\hat{A} = \mathrm{midrange}(Y_1, Y_2, \ldots, Y_n) - 0.5[\mathrm{range}(Y_1, Y_2, \ldots, Y_n)] = Y_1$$

$$\hat{B} = \mathrm{midrange}(Y_1, Y_2, \ldots, Y_n) + 0.5[\mathrm{range}(Y_1, Y_2, \ldots, Y_n)] = Y_n$$

Comments

The uniform distribution defines equal probability over a given range for a continuous distribution. For this reason, it is important as a reference distribution.

One of the most important applications of the uniform distribution is in the generation of random numbers. That is, almost all random number generators generate random numbers on the (0,1) interval. For other distributions, some transformation is applied to the uniform random numbers.

Software

Most general purpose statistical software programs support at least some of the probability functions for the uniform distribution.

1.3.6.6.3. Cauchy Distribution

Probability Density Function

The general formula for the probability density function of the Cauchy distribution is

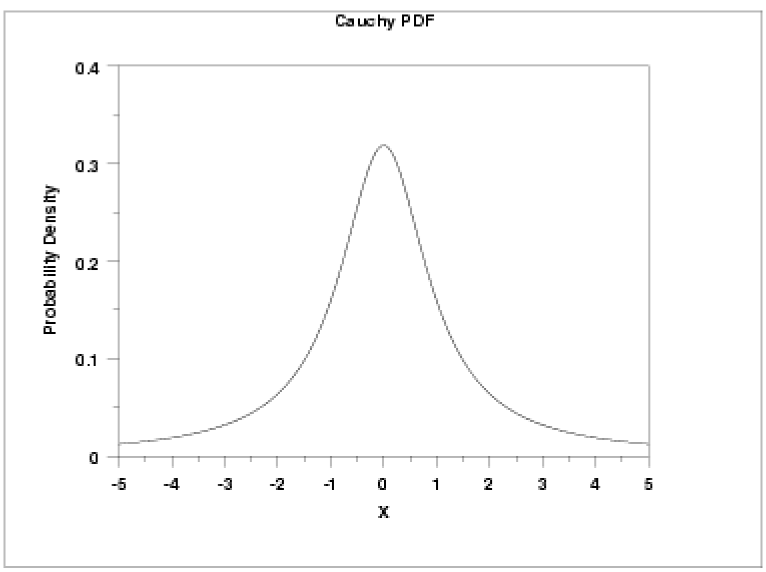
$$f(x) = \frac{1}{s\pi(1 + ((x - t)/s)^2)}$$

where ***t*** is the location parameter and ***s*** is the scale parameter. The case where ***t*** = 0 and ***s*** = 1 is called the **standard Cauchy distribution**. The equation for the standard Cauchy distribution reduces to

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the standard Cauchy probability density function.



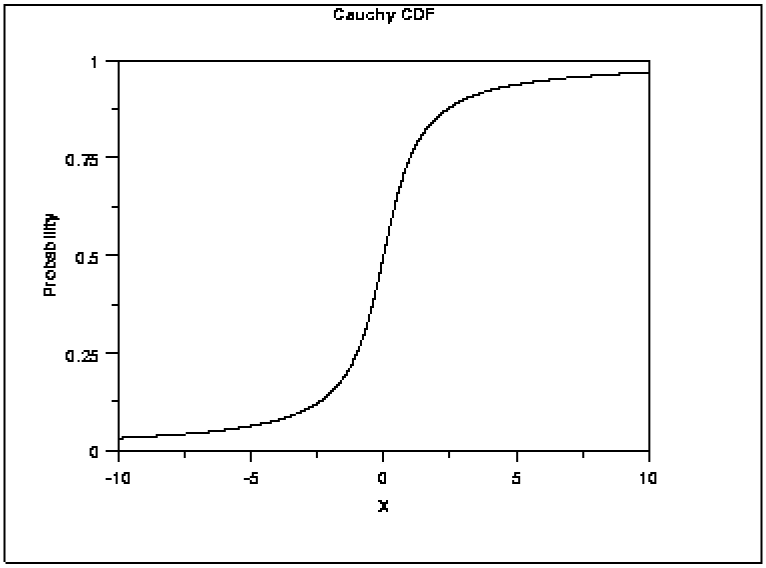
Cumulative Distribution

The formula for the cumulative distribution function for the Cauchy distribution is

Function

$$F(x) = 0.5 + \frac{\arctan(x)}{\pi}$$

The following is the plot of the Cauchy cumulative distribution function.

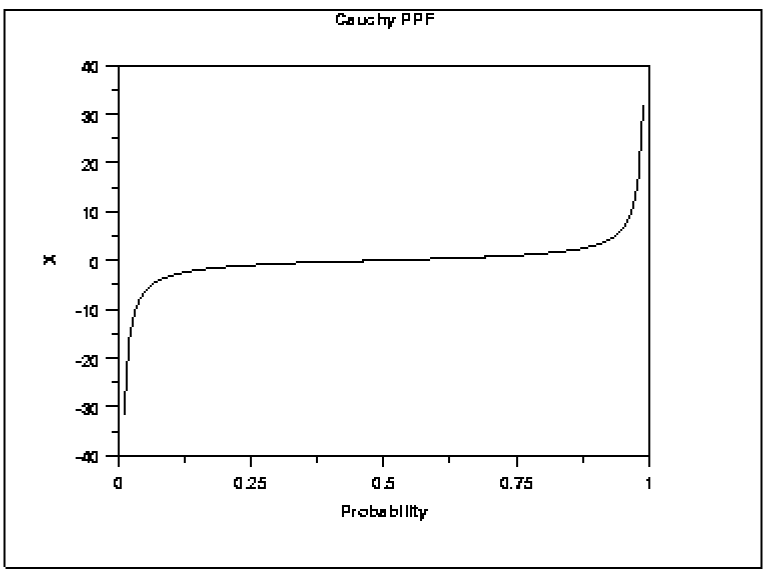


Percent Point Function

The formula for the percent point function of the Cauchy distribution is

$$G(p) = -\cot(\pi p)$$

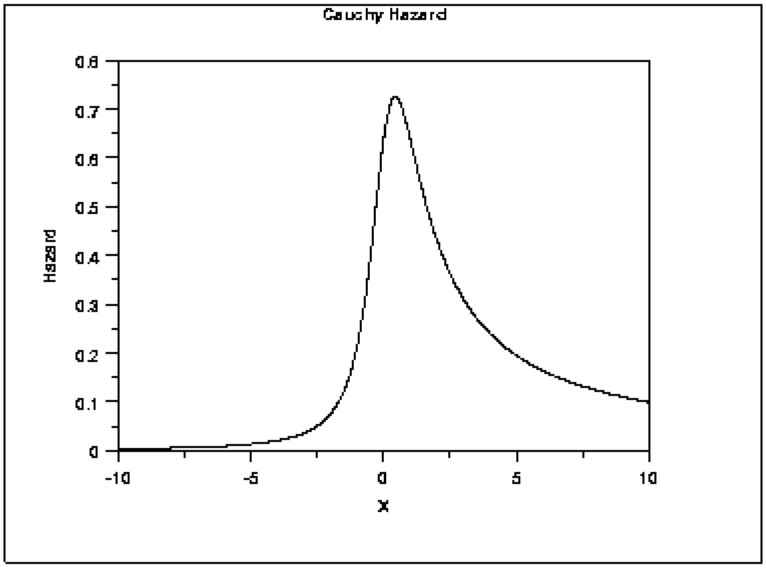
The following is the plot of the Cauchy percent point function.



Hazard Function

The Cauchy hazard function can be computed from the Cauchy probability density and cumulative distribution functions.

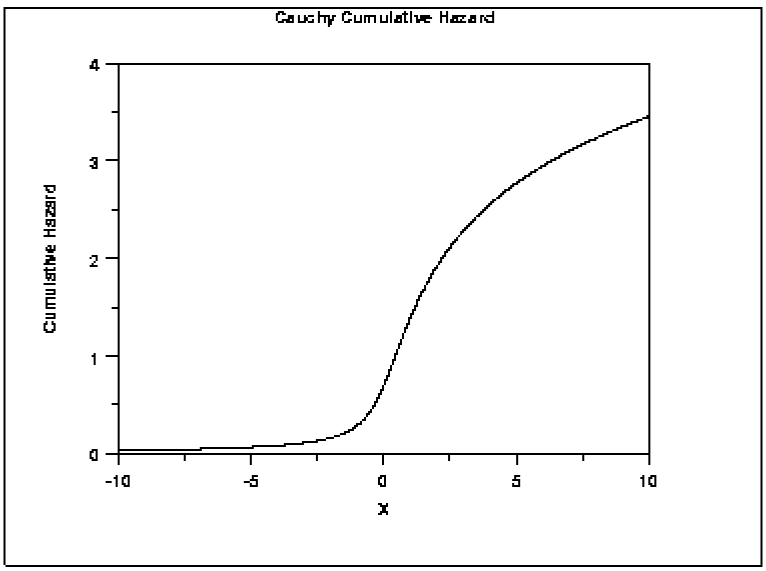
The following is the plot of the Cauchy hazard function.



Cumulative Hazard Function

The Cauchy cumulative hazard function can be computed from the Cauchy cumulative distribution function.

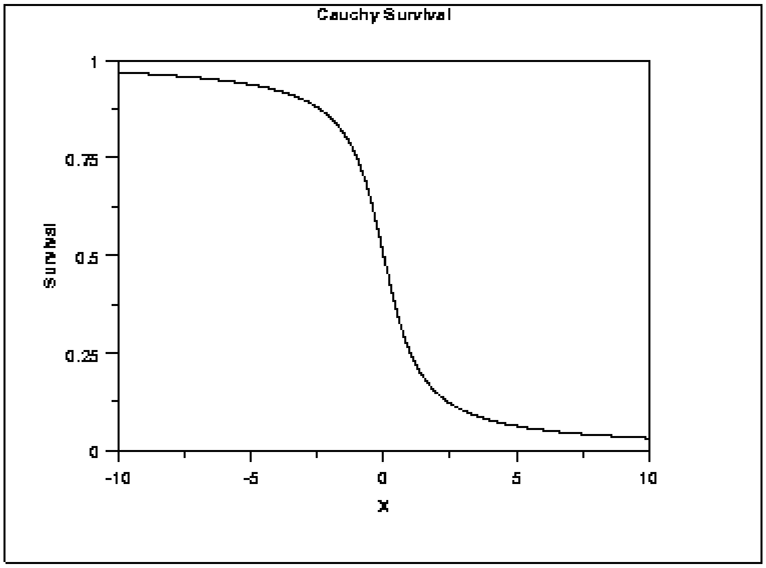
The following is the plot of the Cauchy cumulative hazard function.



Survival Function

The Cauchy survival function can be computed from the Cauchy cumulative distribution function.

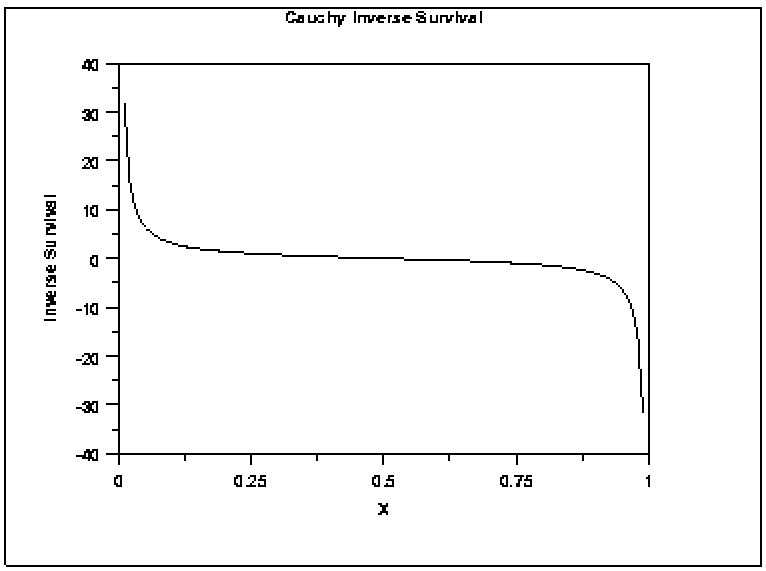
The following is the plot of the Cauchy survival function.



Inverse Survival Function

The Cauchy inverse survival function can be computed from the Cauchy percent point function.

The following is the plot of the Cauchy inverse survival function.



Common Statistics

Mean	The mean is undefined.
Median	The location parameter *t*.
Mode	The location parameter *t*.
Range	Infinity in both directions.
Standard Deviation	The standard deviation is undefined.
Coefficient of Variation	The coefficient of variation is undefined.
Skewness	The skewness is undefined.
Kurtosis	The kurtosis is undefined.

Parameter

The likelihood functions for the Cauchy maximum likelihood

Estimation	estimates are given in chapter 16 of Johnson, Kotz, and Balakrishnan. These equations typically must be solved numerically on a computer.
Comments	The Cauchy distribution is important as an example of a pathological case. Cauchy distributions look similar to a normal distribution. However, they have much heavier tails. When studying hypothesis tests that assume normality, seeing how the tests perform on data from a Cauchy distribution is a good indicator of how sensitive the tests are to heavy-tail departures from normality. Likewise, it is a good check for robust techniques that are designed to work well under a wide variety of distributional assumptions. The mean and standard deviation of the Cauchy distribution are undefined. The practical meaning of this is that collecting 1,000 data points gives no more accurate an estimate of the mean and standard deviation than does a single point.
Software	Many general purpose statistical software programs support at least some of the probability functions for the Cauchy distribution.

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.3.6.6.4. t Distribution

Probability Density Function

The formula for the probability density function of the *t* distribution is

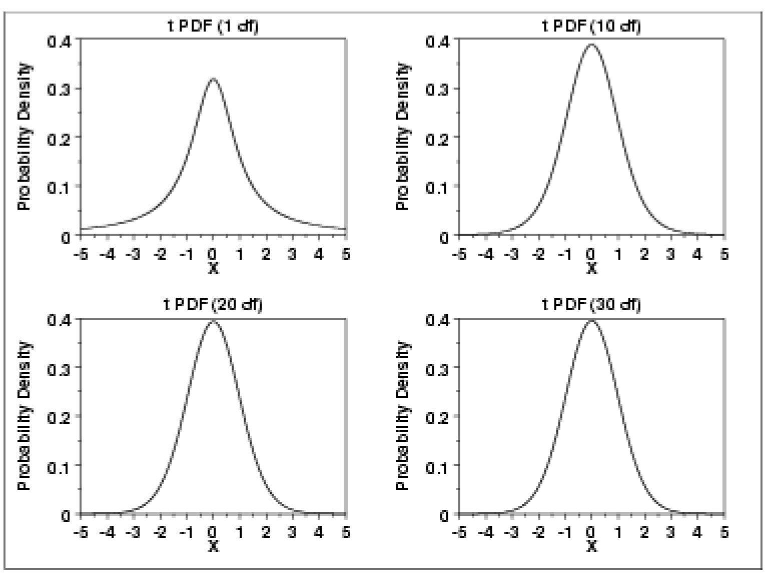
$$f(x) = \frac{\left(1+\frac{x^2}{\nu}\right)^{\frac{-(\nu+1)}{2}}}{B(0.5, 0.5\nu)\sqrt{\nu}}$$

where B is the beta function and ν is a positive integer shape parameter. The formula for the beta function is

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

In a testing context, the *t* distribution is treated as a "standardized distribution" (i.e., no location or scale parameters). However, in a distributional modeling context (as with other probability distributions), the *t* distribution itself can be transformed with a location parameter, μ, and a scale parameter, σ.

The following is the plot of the *t* probability density function for 4 different values of the shape parameter.

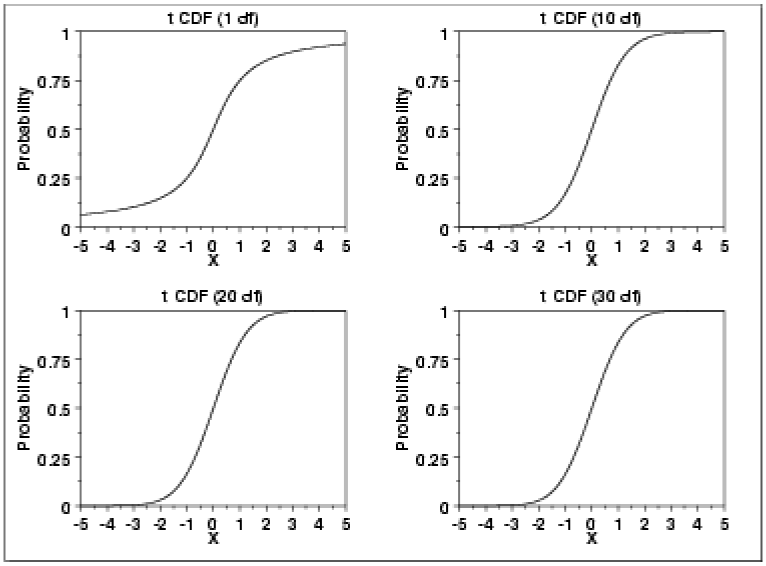


These plots all have a similar shape. The difference is in the heaviness of the tails. In fact, the *t* distribution with ν equal to 1 is a Cauchy distribution. The t distribution approaches a normal distribution as ν becomes large. The approximation is quite good for values of $\nu > 30$.

Cumulative Distribution Function

The formula for the cumulative distribution function of the t distribution is complicated and is not included here. It is given in the Evans, Hastings, and Peacock book.

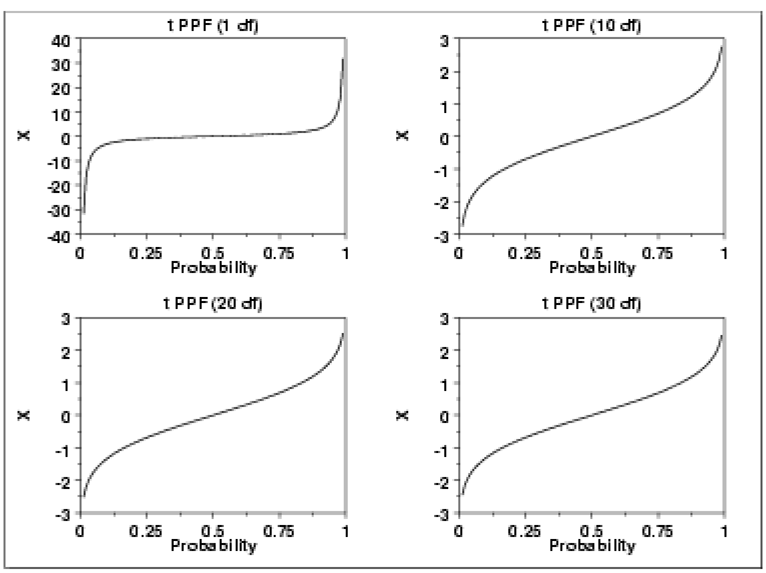
The following are the plots of the t cumulative distribution function with the same values of ν as the pdf plots above.



Percent Point Function

The formula for the percent point function of the t distribution does not exist in a simple closed form. It is computed numerically.

The following are the plots of the t percent point function with the same values of ν as the pdf plots above.



Other Probability Functions

Since the t distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit the formulas and plots for the hazard, cumulative hazard, survival, and inverse survival probability functions.

Common Statistics

Mean	0 (It is undefined for ν equal to 1.)
Median	0
Mode	0
Range	Infinity in both directions.
Standard Deviation	$\sqrt{\frac{\nu}{(\nu-2)}}$ It is undefined for ν equal to 1 or 2.
Coefficient of Variation	Undefined
Skewness	0. It is undefined for ν less than or equal to 3. However, the t distribution is symmetric in all cases.
Kurtosis	$\frac{3(\nu-2)}{(\nu-4)}$ It is undefined for ν less than or equal to 4.

Parameter Estimation

Since the *t* distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit any discussion of parameter estimation.

Comments

The *t* distribution is used in many cases for the critical regions for hypothesis tests and in determining confidence intervals. The most common example is testing if data are consistent with the assumed process mean.

Software

Most general purpose statistical software programs support at least some of the probability functions for the *t* distribution.

1.3.6.6.5. F Distribution

Probability Density Function

The F distribution is the ratio of two chi-square distributions with degrees of freedom ν_1 and ν_2, respectively, where each chi-square has first been divided by its degrees of freedom. The formula for the probability density function of the F distribution is

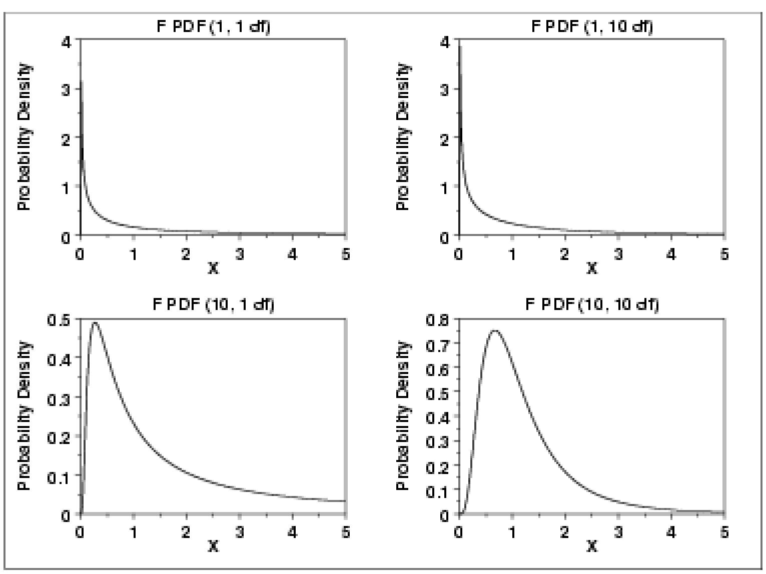
$$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\frac{\nu_1}{\nu_2})^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2}-1}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})(1+\frac{\nu_1 x}{\nu_2})^{\frac{\nu_1+\nu_2}{2}}}$$

where ν_1 and ν_2 are the shape parameters and Γ is the gamma function. The formula for the gamma function is

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$$

In a testing context, the F distribution is treated as a "standardized distribution" (i.e., no location or scale parameters). However, in a distributional modeling context (as with other probability distributions), the F distribution itself can be transformed with a location parameter, μ, and a scale parameter, σ.

The following is the plot of the F probability density function for 4 different values of the shape parameters.



Cumulative Distribution Function

The formula for the Cumulative distribution function of the F distribution is

$$F(x) = 1 - I_k(\frac{\nu_2}{2}, \frac{\nu_1}{2})$$

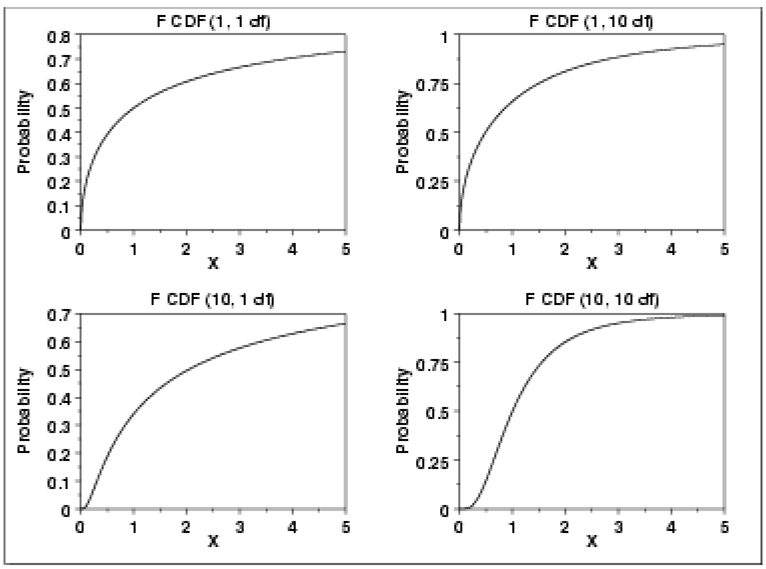
where k = ν_2/ (ν_2 + ν_1*x) and I_k is the incomplete beta function. The formula for the incomplete beta function is

$$I_k(x, \alpha, \beta) = \frac{\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt}{B(\alpha, \beta)}$$

where B is the beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

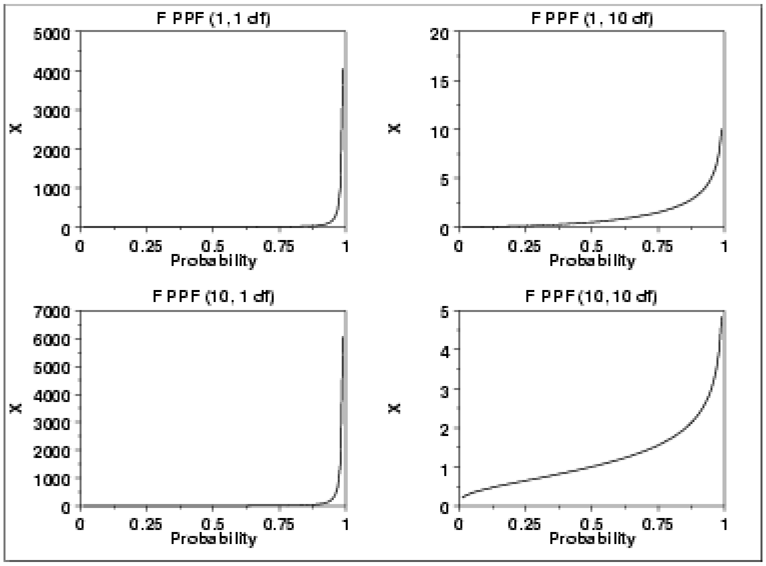
The following is the plot of the F cumulative distribution function with the same values of ν_1 and ν_2 as the pdf plots above.



Percent Point Function

The formula for the percent point function of the F distribution does not exist in a simple closed form. It is computed numerically.

The following is the plot of the F percent point function with the same values of ν_1 and ν_2 as the pdf plots above.



Other Probability Functions	Since the F distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit the formulas and plots for the hazard, cumulative hazard, survival, and inverse survival probability functions.

Common Statistics

The formulas below are for the case where the location parameter is zero and the scale parameter is one.

Mean	$\frac{\nu_2}{(\nu_2 - 2)}$	$\nu_2 > 2$
Mode	$\frac{\nu_2(\nu_1 - 2)}{\nu_1(\nu_2 + 2)}$	$\nu_1 > 2$
Range	0 to positive infinity	
Standard Deviation	$\sqrt{\frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}}$	$\nu_2 > 4$
Coefficient of Variation	$\sqrt{\frac{2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 4)}}$	$\nu_2 > 4$
Skewness	$\frac{(2\nu_1 + \nu_2 - 2)\sqrt{8(\nu_2 - 4)}}{\sqrt{\nu_1}(\nu_2 - 6)\sqrt{(\nu_1 + \nu_2 - 2)}}$	$\nu_2 > 6$

Parameter Estimation	Since the F distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit any discussion of parameter estimation.
Comments	The F distribution is used in many cases for the critical regions for hypothesis tests and in determining confidence intervals. Two common examples are the analysis of variance and the F test to determine if the variances of two populations are equal.

Software Most general purpose statistical software programs support at least some of the probability functions for the F distribution.

1.3.6.6.6. Chi-Square Distribution

Probability Density Function

The chi-square distribution results when ν independent variables with standard normal distributions are squared and summed. The formula for the probability density function of the chi-square distribution is

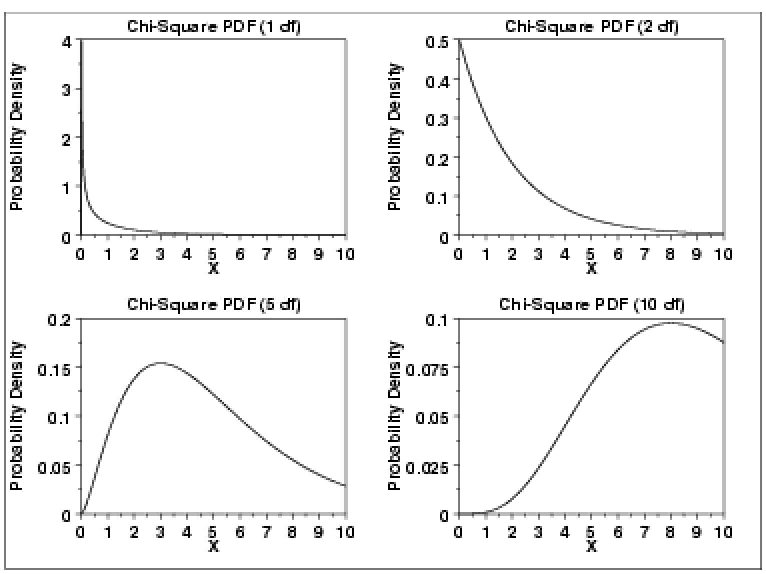
$$f(x) = \frac{e^{\frac{-x}{2}} x^{\frac{\nu}{2}-1}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \qquad \mbox{for } x \ge 0$$

where ν is the shape parameter and Γ is the gamma function. The formula for the gamma function is

$$\Gamma(a) = \int_{0}^{\infty} t^{a-1}e^{-t}dt$$

In a testing context, the chi-square distribution is treated as a "standardized distribution" (i.e., no location or scale parameters). However, in a distributional modeling context (as with other probability distributions), the chi-square distribution itself can be transformed with a location parameter, μ, and a scale parameter, σ.

The following is the plot of the chi-square probability density function for 4 different values of the shape parameter.



Cumulative Distribution Function

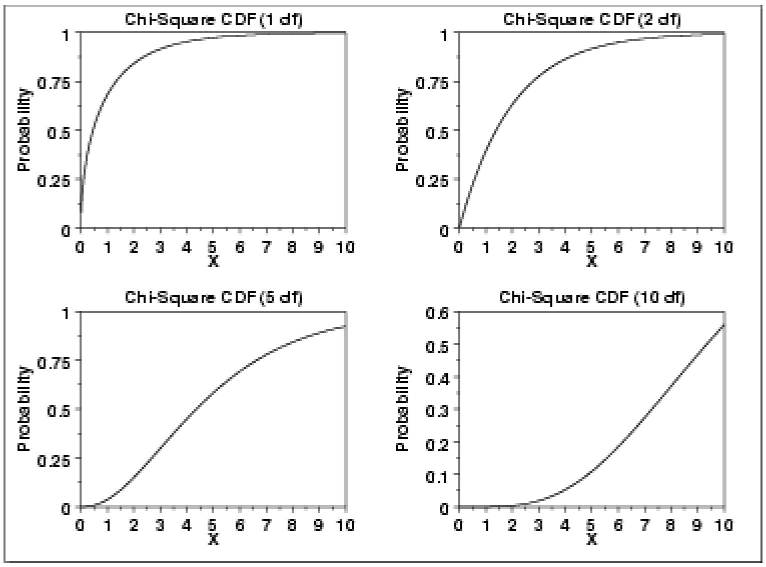
The formula for the cumulative distribution function of the chi-square distribution is

$$F(x) = \frac{\gamma(\frac{\nu}{2}, \frac{x}{2})}{\Gamma(\frac{\nu}{2})} \qquad \text{for } x \ge 0$$

where Γ is the gamma function defined above and γ is the incomplete gamma function. The formula for the incomplete gamma function is

$$\Gamma_x(a) = \int_0^x t^{a-1} e^{-t} dt$$

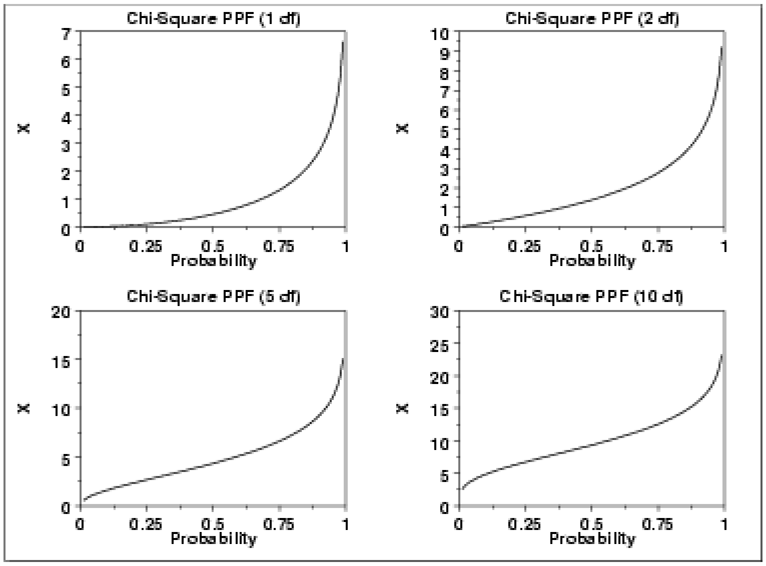
The following is the plot of the chi-square cumulative distribution function with the same values of ν as the pdf plots above.



Percent Point Function

The formula for the percent point function of the chi-square distribution does not exist in a simple closed form. It is computed numerically.

The following is the plot of the chi-square percent point function with the same values of ν as the pdf plots above.



Other Probability Functions	Since the chi-square distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit the formulas and plots for the hazard, cumulative hazard, survival, and inverse survival probability functions.

Common Statistics

Mean	ν
Median	approximately ν - 2/3 for large ν
Mode	$\nu - 2 \quad \text{for } \nu > 2$
Range	0 to positive infinity
Standard Deviation	$\sqrt{2\nu}$
Coefficient of Variation	$\sqrt{\frac{2}{\nu}}$
Skewness	$\frac{2^{1.5}}{\sqrt{\nu}}$
Kurtosis	$3 + \frac{12}{\nu}$

Parameter Estimation

Since the chi-square distribution is typically used to develop hypothesis tests and confidence intervals and rarely for modeling applications, we omit any discussion of parameter estimation.

Comments

The chi-square distribution is used in many cases for the critical regions for hypothesis tests and in determining confidence intervals. Two common examples are the chi-square test for independence in an ***RxC*** contingency table and the chi-square test to determine if the standard deviation of a population is equal to a pre-specified value.

Software

Most general purpose statistical software programs support at least some of the probability functions for the chi-square distribution.

1.3.6.6.7. Exponential Distribution

Probability Density Function

The general formula for the probability density function of the exponential distribution is

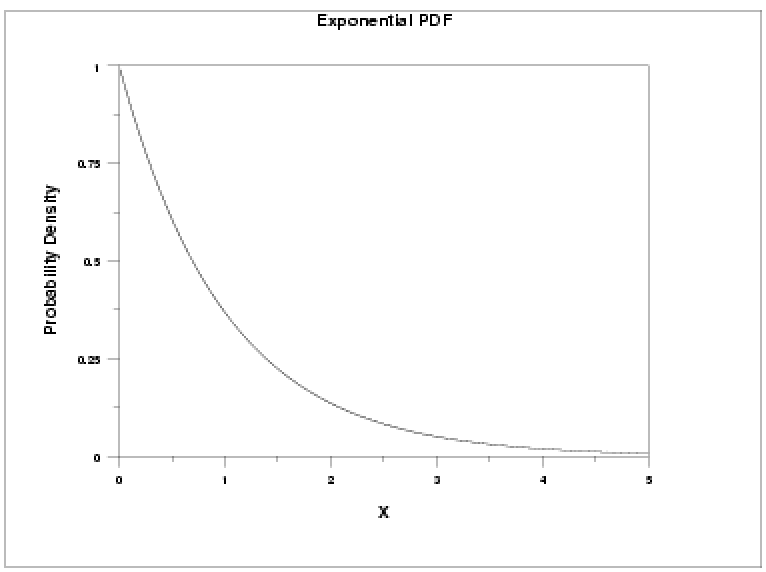
$$f(x) = \frac{1}{\beta} e^{-(x-\mu)/\beta} \qquad x \ge \mu; \beta > 0$$

where μ is the location parameter and β is the scale parameter (the scale parameter is often referred to as λ which equals $1/\beta$). The case where μ = 0 and β = 1 is called the **standard exponential distribution**. The equation for the standard exponential distribution is

$$f(x) = e^{-x} \qquad \mathrm{for}\ x \ge 0$$

The general form of probability functions can be expressed in terms of the standard distribution. Subsequent formulas in this section are given for the 1-parameter (i.e., with scale parameter) form of the function.

The following is the plot of the exponential probability density function.



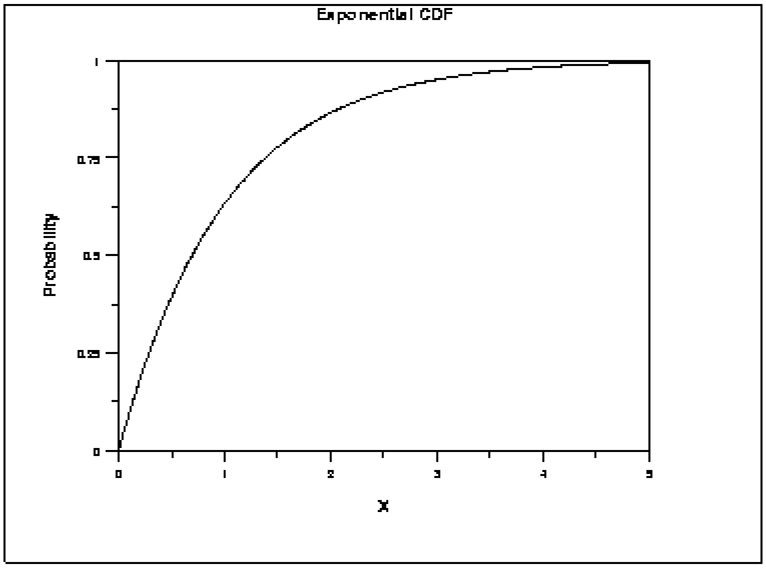
Cumulative Distribution

The formula for the cumulative distribution function of the exponential distribution is

Function

$$F(x) = 1 - e^{-x/\beta} \qquad x \ge 0; \beta > 0$$

The following is the plot of the exponential cumulative distribution function.

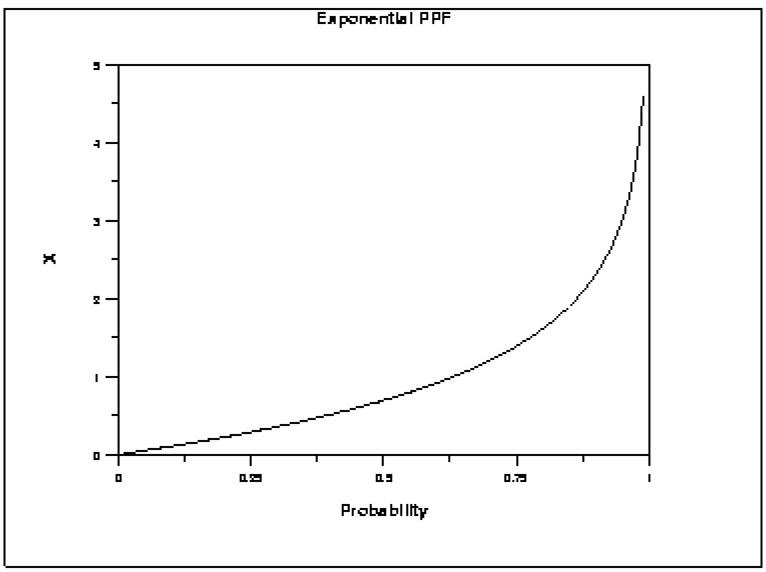


Percent Point Function

The formula for the percent point function of the exponential distribution is

$$G(p) = -\beta \ln(1 - p) \qquad 0 \le p < 1; \beta > 0$$

The following is the plot of the exponential percent point function.

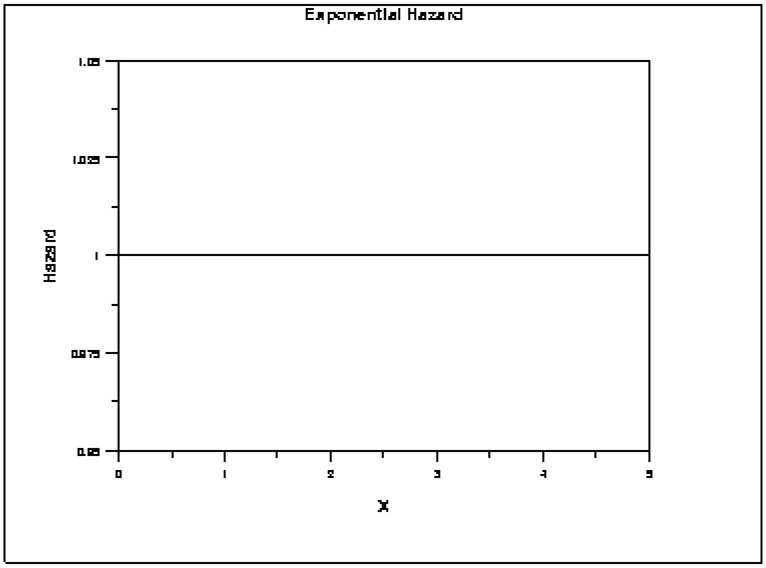


Hazard Function

The formula for the hazard function of the exponential distribution is

$$h(x) = \frac{1}{\beta} \qquad x \ge 0; \beta > 0$$

The following is the plot of the exponential hazard function.

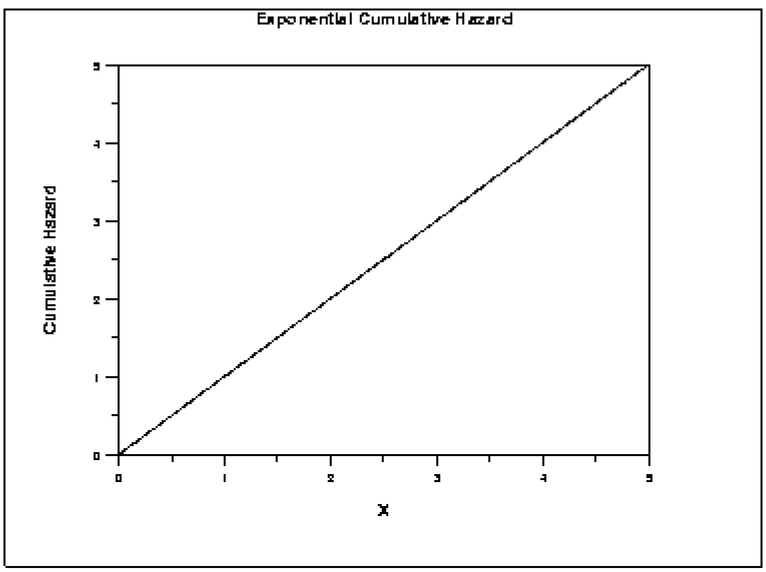


Cumulative Hazard Function

The formula for the cumulative hazard function of the exponential distribution is

$$H(x) = \frac{x}{\beta} \qquad x \ge 0; \beta > 0$$

The following is the plot of the exponential cumulative hazard function.

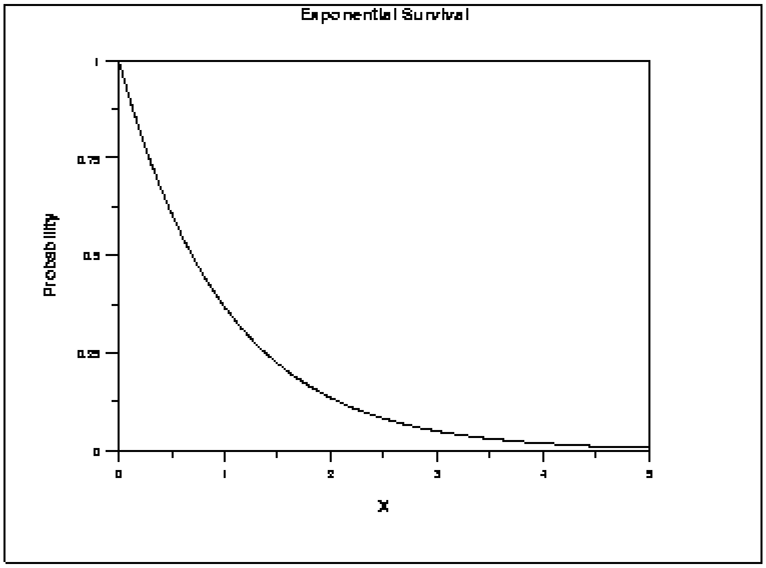


Survival Function

The formula for the survival function of the exponential distribution is

$$S(x) = e^{-x/\beta} \qquad x \ge 0; \beta > 0$$

The following is the plot of the exponential survival function.

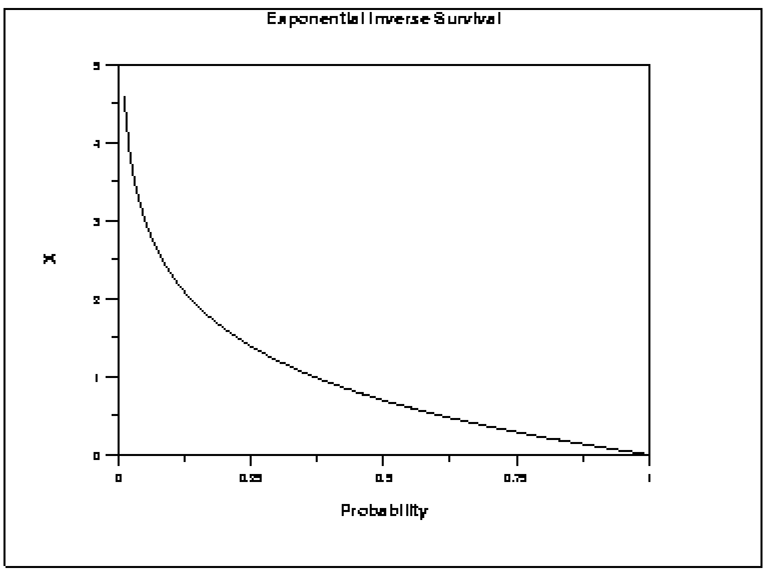


Inverse Survival Function

The formula for the inverse survival function of the exponential distribution is

$$Z(p) = -\beta \ln(p) \qquad 0 \le p < 1; \beta > 0$$

The following is the plot of the exponential inverse survival function.



Common Statistics

Mean	β
Median	$\beta \ln 2$
Mode	Zero
Range	Zero to plus infinity
Standard Deviation	β
Coefficient of Variation	1
Skewness	2
Kurtosis	9

Parameter Estimation	For the full sample case, the maximum likelihood estimator of the scale parameter is the sample mean. Maximum likelihood estimation for the exponential distribution is discussed in the chapter on reliability (Chapter 8). It is also discussed in chapter 19 of Johnson, Kotz, and Balakrishnan.
Comments	The exponential distribution is primarily used in reliability applications. The exponential distribution is used to model data with a constant failure rate (indicated by the hazard plot which is simply equal to a constant).
Software	Most general purpose statistical software programs support at least some of the probability functions for the exponential distribution.

1.3.6.6.8. Weibull Distribution

Probability Density Function

The formula for the probability density function of the general Weibull distribution is

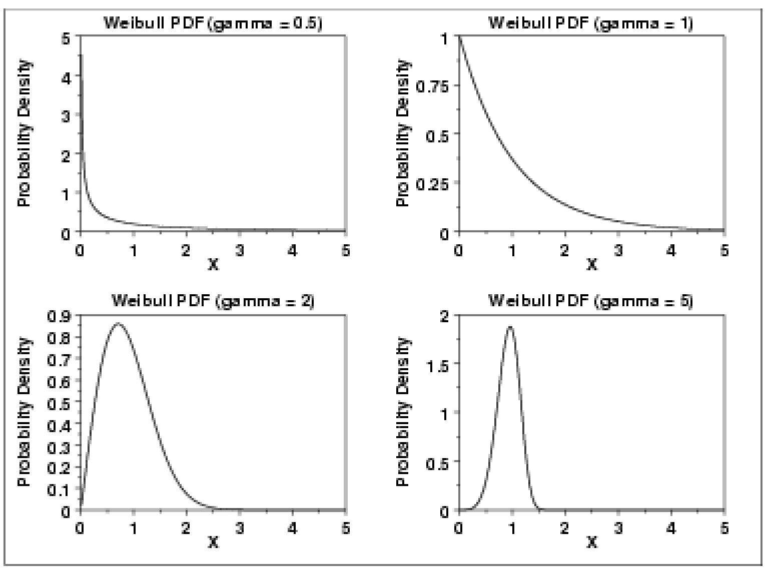
$$f(x) = \frac{\gamma}{\alpha}\left(\frac{x-\mu}{\alpha}\right)^{(\gamma-1)}\exp\left(-((x-\mu)/\alpha)^{\gamma}\right) \qquad x \ge \mu; \gamma, \alpha > 0$$

where γ is the shape parameter, μ is the location parameter and α is the scale parameter. The case where μ = 0 and α = 1 is called the **standard Weibull distribution**. The case where μ = 0 is called the 2-parameter Weibull distribution. The equation for the standard Weibull distribution reduces to

$$f(x) = \gamma x^{(\gamma-1)}\exp(-(x^{\gamma})) \qquad x \ge 0; \gamma > 0$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the Weibull probability density function.

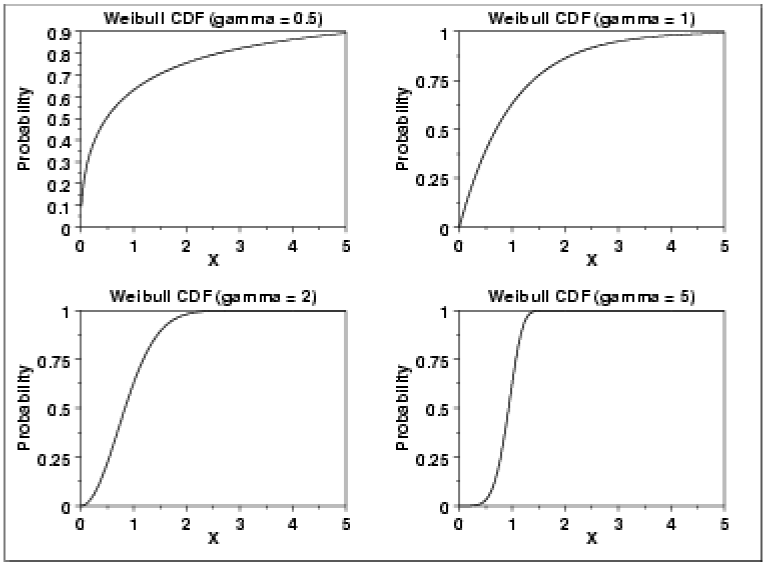


Cumulative Distribution Function

The formula for the cumulative distribution function of the Weibull distribution is

$$F(x) = 1 - e^{-(x^{\gamma})} \qquad x \ge 0; \gamma > 0$$

The following is the plot of the Weibull cumulative distribution function with the same values of γ as the pdf plots above.

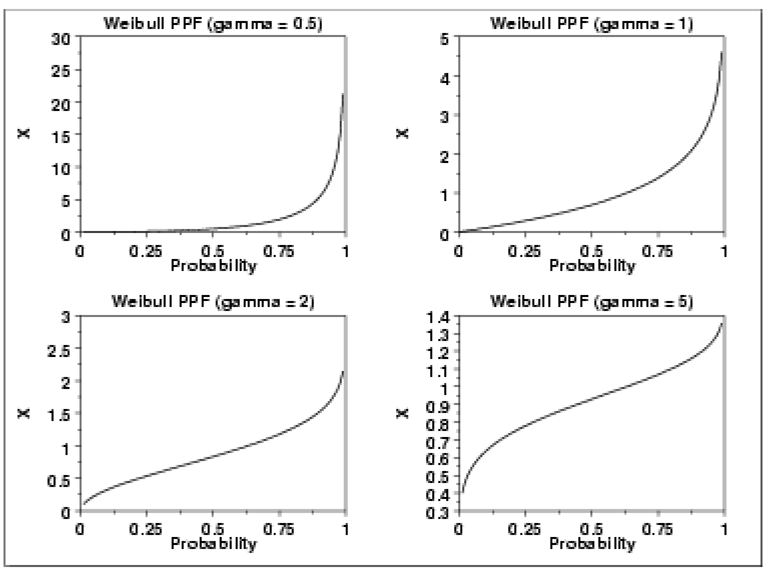


Percent Point Function

The formula for the [percent point function](https://www.itl.nist.gov/div898/handbook/eda/section3/eda362.htm#PPF) of the Weibull distribution is

$$G(p) = (-\ln(1-p))^{1/\gamma} \qquad 0 \le p < 1; \gamma > 0$$

The following is the plot of the Weibull percent point function with the same values of γ as the pdf plots above.

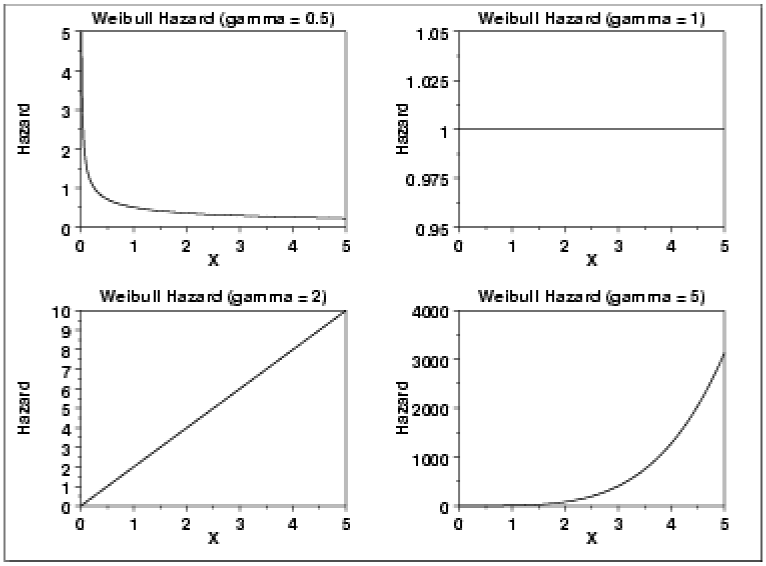


Hazard Function

The formula for the [hazard function](https://www.itl.nist.gov/div898/handbook/eda/section3/eda362.htm#HAZ) of the Weibull distribution is

$$h(x) = \gamma x^{(\gamma-1)} \qquad x \ge 0; \gamma > 0$$

The following is the plot of the Weibull hazard function with the same values of γ as the pdf plots above.

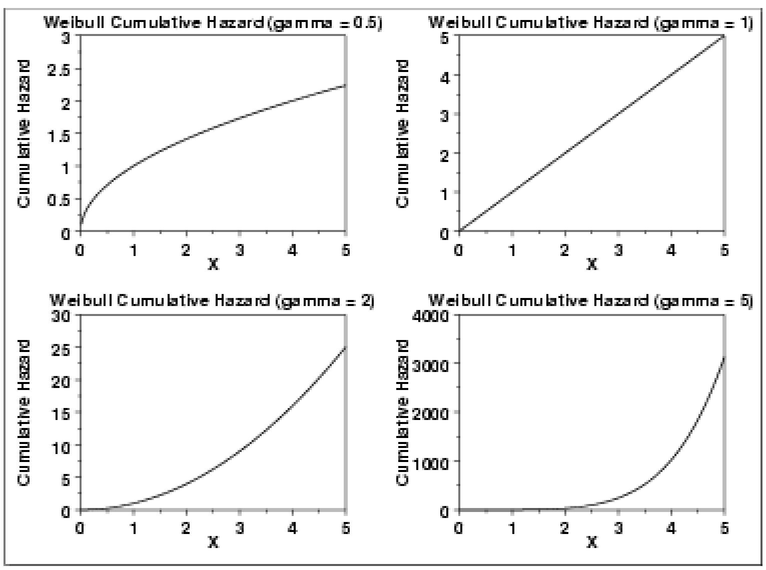


Cumulative Hazard Function

The formula for the cumulative hazard function of the Weibull distribution is

$$H(x) = x^{\gamma} \qquad x \ge 0; \gamma > 0$$

The following is the plot of the Weibull cumulative hazard function with the same values of γ as the pdf plots above.

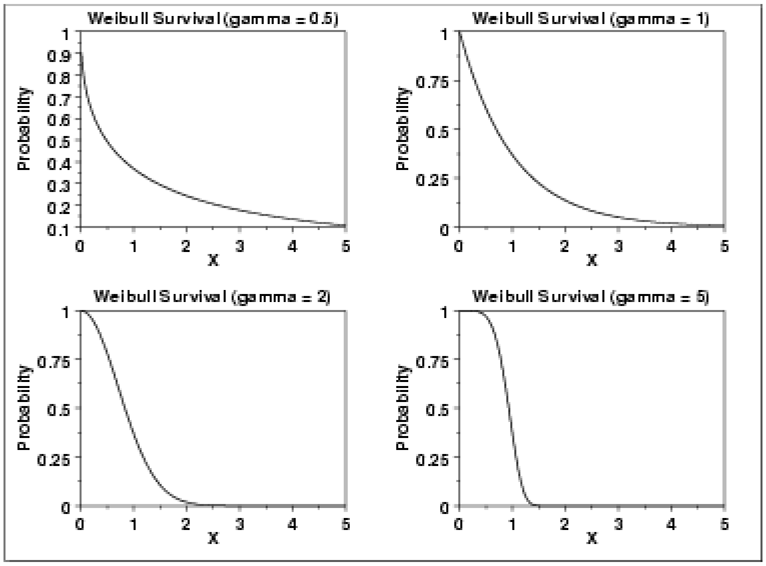


Survival Function

The formula for the survival function of the Weibull distribution is

$$S(x) = \exp -(x^{\gamma}) \qquad x \ge 0; \gamma > 0$$

The following is the plot of the Weibull survival function with the same values of γ as the pdf plots above.

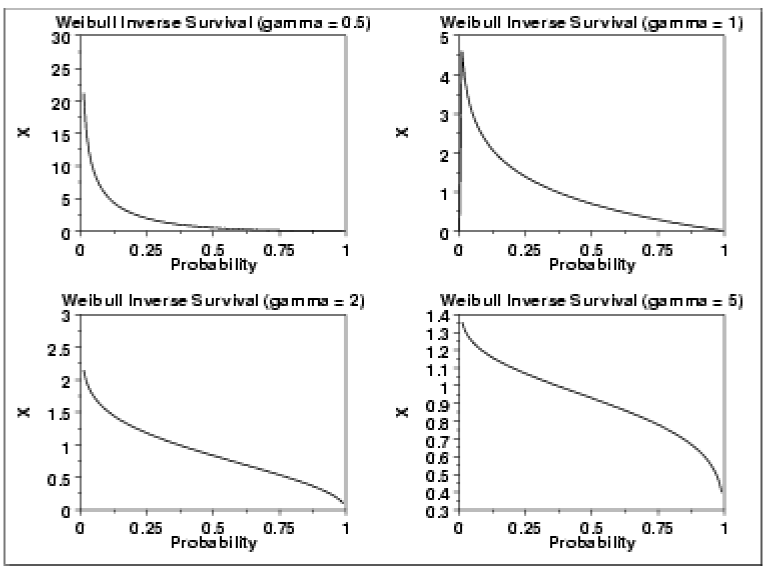


Inverse Survival Function

The formula for the inverse survival function of the Weibull distribution is

$$Z(p) = (-\ln(p))^{1/\gamma} \qquad 0 \le p < 1; \gamma > 0$$

The following is the plot of the Weibull inverse survival function with the same values of γ as the pdf plots above.



Common Statistics

The formulas below are with the location parameter equal to zero and the scale parameter equal to one.

Mean

$$\Gamma(\frac{\gamma+1}{\gamma})$$

where Γ is the gamma function

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$$

Median

$$\ln(2)^{1/\gamma}$$

Mode

	$(1-\frac{1}{\gamma})^{1/\gamma} \quad \gamma > 1$ $0 \quad \gamma \le 1$
Range	Zero to positive infinity.
Standard Deviation	$\sqrt{\Gamma(\frac{\gamma+2}{\gamma}) - (\Gamma(\frac{\gamma+1}{\gamma}))^2}$
Coefficient of Variation	$\sqrt{\frac{\Gamma(\frac{\gamma+2}{\gamma})}{(\Gamma(\frac{\gamma+1}{\gamma}))^2} - 1}$

Parameter Estimation Maximum likelihood estimation for the Weibull distribution is discussed in the Reliability chapter (Chapter 8). It is also discussed in Chapter 21 of Johnson, Kotz, and Balakrishnan.

Comments The Weibull distribution is used extensively in reliability applications to model failure times.

Software Most general purpose statistical software programs support at least some of the probability functions for the Weibull distribution.

1.3.6.6.9. Lognormal Distribution

Probability Density Function

A variable X is lognormally distributed if Y = LN(X) is normally distributed with "LN" denoting the natural logarithm. The general formula for the probability density function of the lognormal distribution is

$$f(x) = \frac{e^{-((\ln((x-\theta)/m))^2/(2\sigma^2))}}{(x-\theta)\sigma\sqrt{2\pi}} \qquad x \ge \theta; m, \sigma > 0$$

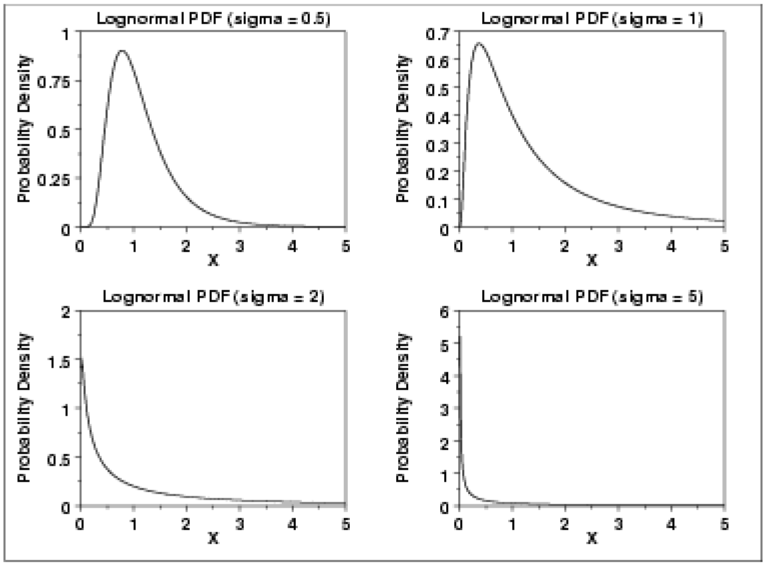
where σ is the shape parameter, θ is the location parameter and m is the scale parameter. The case where θ = 0 and m = 1 is called the **standard lognormal distribution**. The case where θ equals zero is called the 2-parameter lognormal distribution.

The equation for the standard lognormal distribution is

$$f(x) = \frac{e^{-((\ln x)^2/2\sigma^2)}}{x\sigma\sqrt{2\pi}} \qquad x \ge 0; \sigma > 0$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the lognormal probability density function for four values of σ.



There are several common parameterizations of the lognormal distribution. The form given here is from Evans, Hastings, and Peacock.

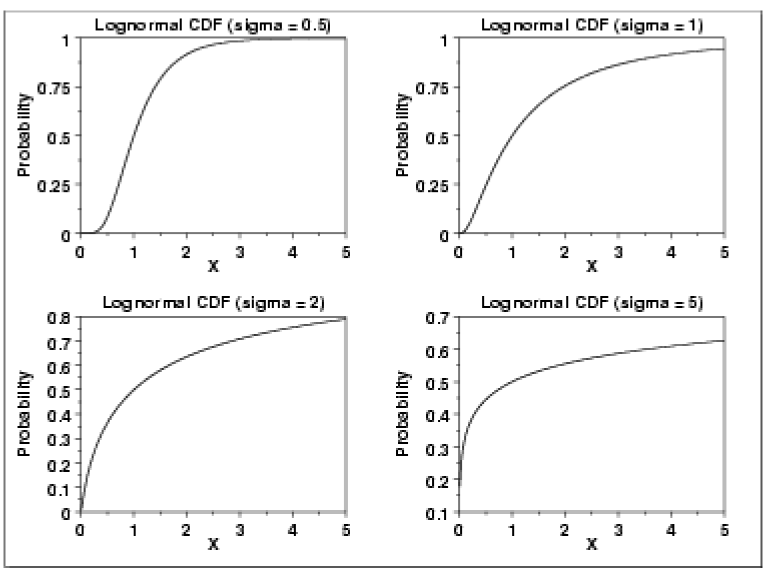
Cumulative Distribution Function

The formula for the cumulative distribution function of the lognormal distribution is

$$F(x) = \Phi\left(\frac{\ln(x)}{\sigma}\right) \qquad x \ge 0; \sigma > 0$$

where Φ is the cumulative distribution function of the normal distribution.

The following is the plot of the lognormal cumulative distribution function with the same values of σ as the pdf plots above.



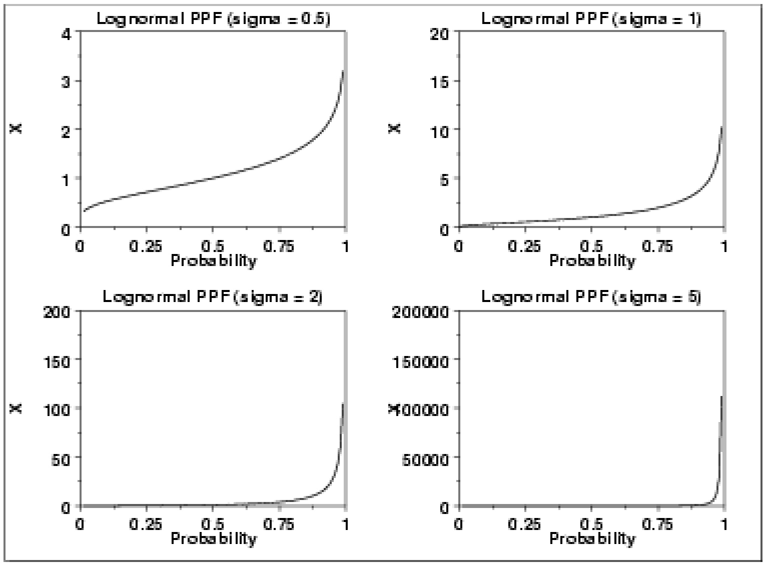
Percent Point Function

The formula for the percent point function of the lognormal distribution is

$$G(p) = \exp(\sigma\Phi^{-1}(p)) \qquad 0 \le p < 1; \sigma > 0$$

where Φ^{-1} is the percent point function of the normal distribution.

The following is the plot of the lognormal percent point function with the same values of σ as the pdf plots above.



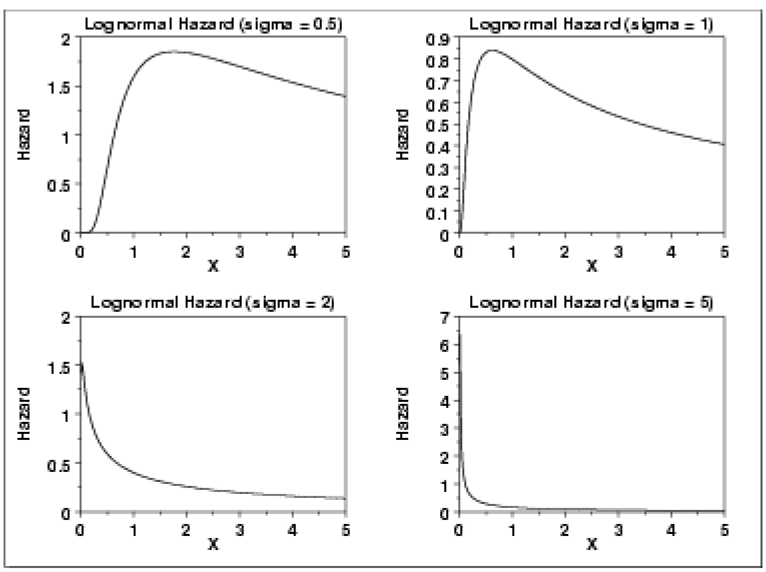
Hazard Function

The formula for the hazard function of the lognormal distribution is

$$h(x,\sigma) = \frac{\left(\frac{1}{x\sigma}\right)\phi\left(\frac{\ln x}{\sigma}\right)}{\Phi\left(\frac{-\ln x}{\sigma}\right)} \qquad x > 0; \sigma > 0$$

where ϕ is the probability density function of the normal distribution and Φ is the cumulative distribution function of the normal distribution.

The following is the plot of the lognormal hazard function with the same values of σ as the pdf plots above.



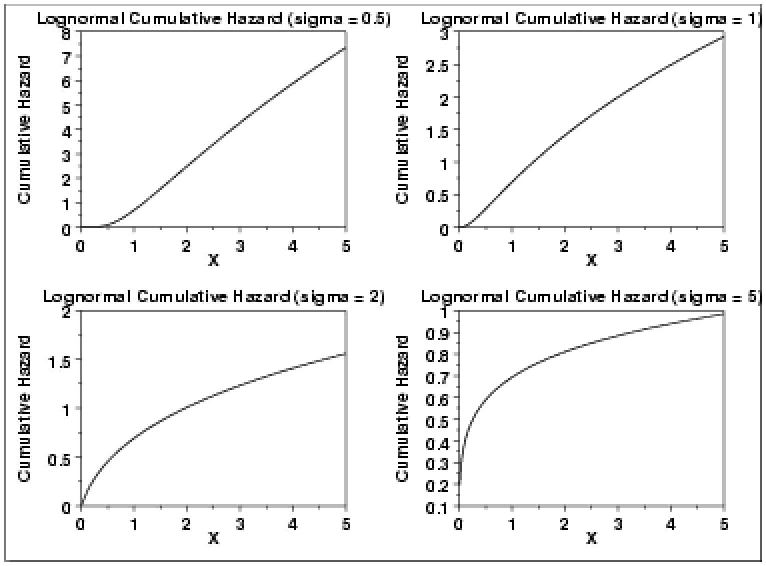
Cumulative Hazard Function

The formula for the cumulative hazard function of the lognormal distribution is

$$H(x) = -\ln(1 - \Phi(\frac{\ln(x)}{\sigma})) \qquad x \ge 0; \sigma > 0$$

where Φ is the cumulative distribution function of the normal distribution.

The following is the plot of the lognormal cumulative hazard function with the same values of σ as the pdf plots above.



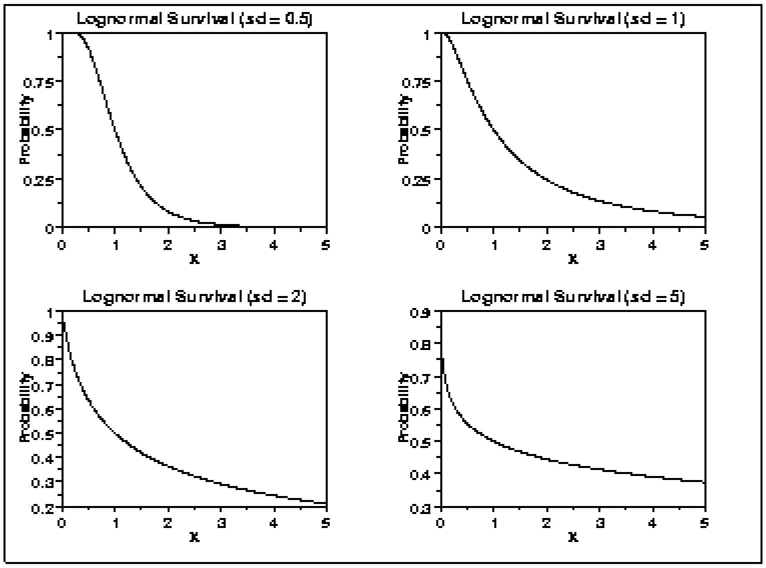
Survival Function

The formula for the survival function of the lognormal distribution is

$$S(x) = 1 - \Phi(\frac{\ln(x)}{\sigma}) \qquad x \ge 0; \sigma > 0$$

where Φ is the cumulative distribution function of the normal distribution.

The following is the plot of the lognormal survival function with the same values of σ as the pdf plots above.



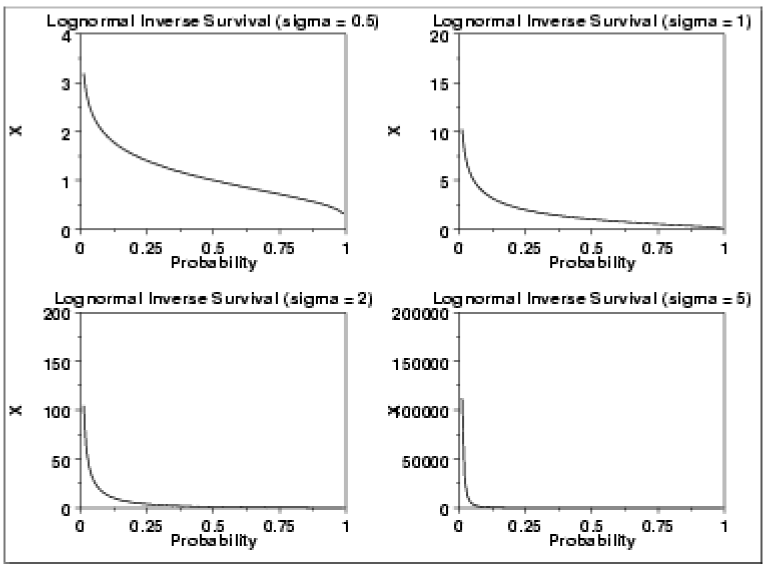
Inverse Survival Function

The formula for the [inverse survival function](#) of the lognormal distribution is

$$Z(p) = \exp(\sigma\Phi^{-1}(1-p)) \qquad 0 \le p < 1; \sigma > 0$$

where Φ^{-1} is the [percent point function of the normal distribution](#).

The following is the plot of the lognormal inverse survival function with the same values of σ as the pdf plots above.



Common Statistics

The formulas below are with the location parameter equal to zero and the scale parameter equal to one.

Mean	$e^{0.5\sigma^2}$
Median	Scale parameter ***m*** (= 1 if scale parameter not specified).
Mode	$\frac{1}{e^{\sigma^2}}$
Range	Zero to positive infinity

Standard Deviation	$\sqrt{e^{\sigma^2}(e^{\sigma^2}-1)}$
Skewness	$(e^{\sigma^2}+2)\sqrt{e^{\sigma^2}-1}$
Kurtosis	$(e^{\sigma^2})^4+2(e^{\sigma^2})^3+3(e^{\sigma^2})^2-3$
Coefficient of Variation	$\sqrt{e^{\sigma^2}-1}$

Parameter Estimation

The maximum likelihood estimates for the scale parameter, m, and the shape parameter, σ, are

$$\hat{m} = \exp\hat{\mu}$$

and

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{N}(\ln(X_i)-\hat{\mu})^2}{N}}$$

where

$$\hat{\mu} = \frac{\sum_{i=1}^{N}\ln X_i}{N}$$

If the location parameter is known, it can be subtracted from the original data points before computing the maximum likelihood estimates of the shape and scale parameters.

Comments

The lognormal distribution is used extensively in reliability applications to model failure times. The lognormal and Weibull distributions are probably the most commonly used distributions in reliability applications.

Software

Most general purpose statistical software programs support at least some of the probability functions for the lognormal distribution.

1.3.6.6.10. Birnbaum-Saunders (Fatigue Life) Distribution

Probability Density Function

The Birnbaum-Saunders distribution is also commonly known as the fatigue life distribution. There are several alternative formulations of the Birnbaum-Saunders distribution in the literature.

The general formula for the probability density function of the Birnbaum-Saunders distribution is

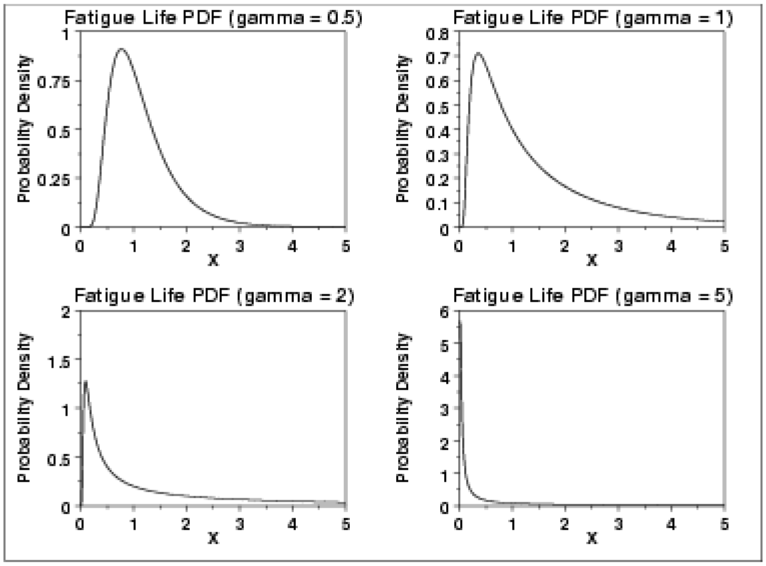
$$f(x) = \left(\frac{\sqrt{\frac{x-\mu}{\beta}} + \sqrt{\frac{\beta}{x-\mu}}}{2\gamma(x-\mu)}\right)\phi\left(\frac{\sqrt{\frac{x-\mu}{\beta}} - \sqrt{\frac{\beta}{x-\mu}}}{\gamma}\right) \qquad x > \mu; \gamma, \beta > 0$$

where γ is the shape parameter, μ is the location parameter, β is the scale parameter, ϕ is the probability density function of the standard normal distribution, and Φ is the cumulative distribution function of the standard normal distribution. The case where μ = 0 and β = 1 is called the **standard Birnbaum-Saunders distribution**. The equation for the standard Birnbaum-Saunders distribution reduces to

$$f(x) = \left(\frac{\sqrt{x} + \sqrt{\frac{1}{x}}}{2\gamma x}\right)\phi\left(\frac{\sqrt{x} - \sqrt{\frac{1}{x}}}{\gamma}\right) \qquad x > 0; \gamma > 0$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the Birnbaum-Saunders probability density function.

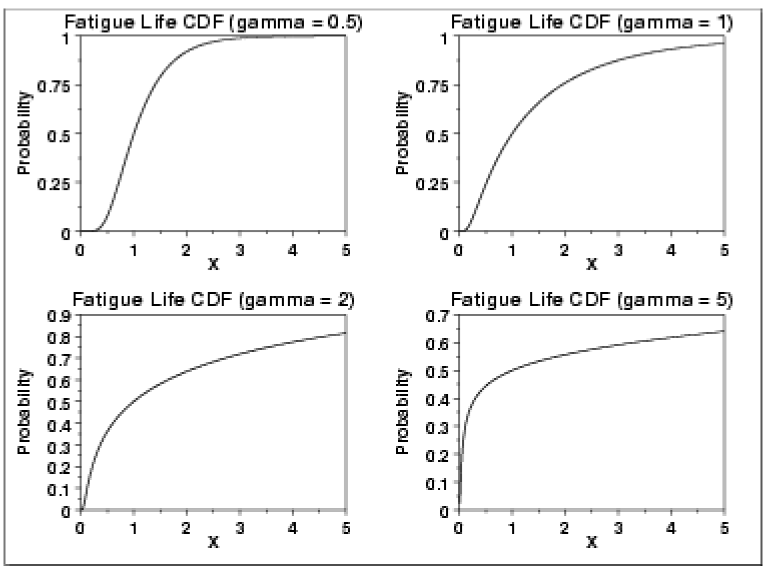


Cumulative Distribution Function

The formula for the cumulative distribution function of the Birnbaum-Saunders distribution is

$$F(x) = \Phi\left(\frac{\sqrt{x} - \sqrt{\frac{1}{x}}}{\gamma}\right) \qquad x > 0; \gamma > 0$$

where Φ is the cumulative distribution function of the standard normal distribution. The following is the plot of the Birnbaum-Saunders cumulative distribution function with the same values of γ as the pdf plots above.

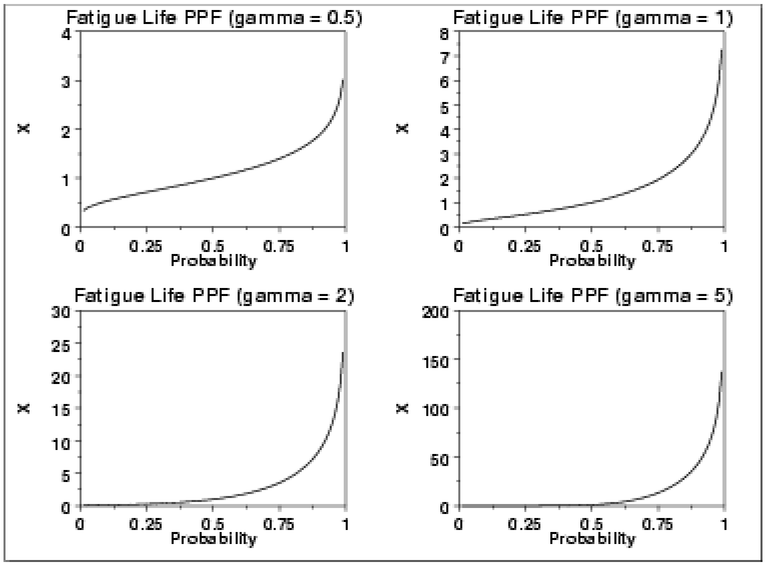


Percent Point Function

The formula for the percent point function of the Birnbaum-Saunders distribution is

$$G(p) = \frac{1}{4}\left[\gamma\Phi^{-1}(p) + \sqrt{4 + (\gamma\Phi^{-1}(p))^2}\right]^2$$

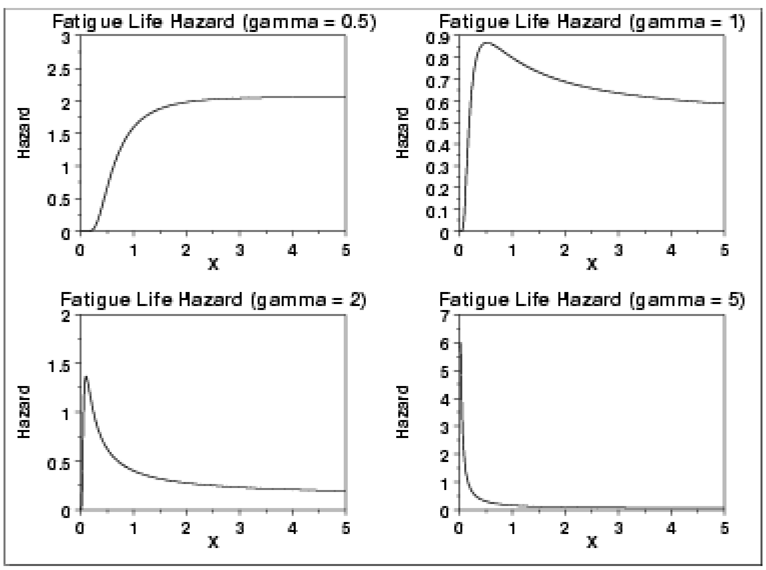
where Φ^{-1} is the percent point function of the standard normal distribution. The following is the plot of the Birnbaum-Saunders percent point function with the same values of γ as the pdf plots above.



Hazard Function

The Birnbaum-Saunders hazard function can be computed from the Birnbaum-Saunders probability density and cumulative distribution functions.

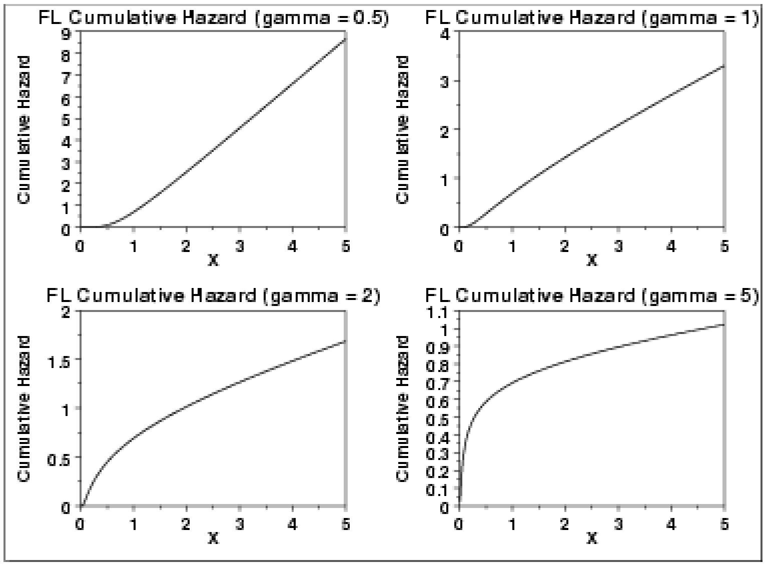
The following is the plot of the Birnbaum-Saunders hazard function with the same values of γ as the pdf plots above.



Cumulative Hazard Function

The Birnbaum-Saunders cumulative hazard function can be computed from the Birnbaum-Saunders cumulative distribution function.

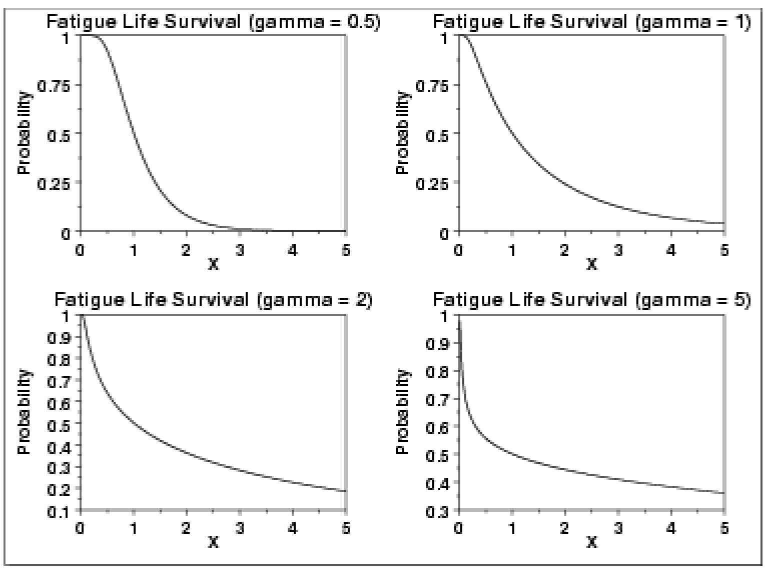
The following is the plot of the Birnbaum-Saunders cumulative hazard function with the same values of γ as the pdf plots above.



Survival Function

The Birnbaum-Saunders survival function can be computed from the Birnbaum-Saunders cumulative distribution function.

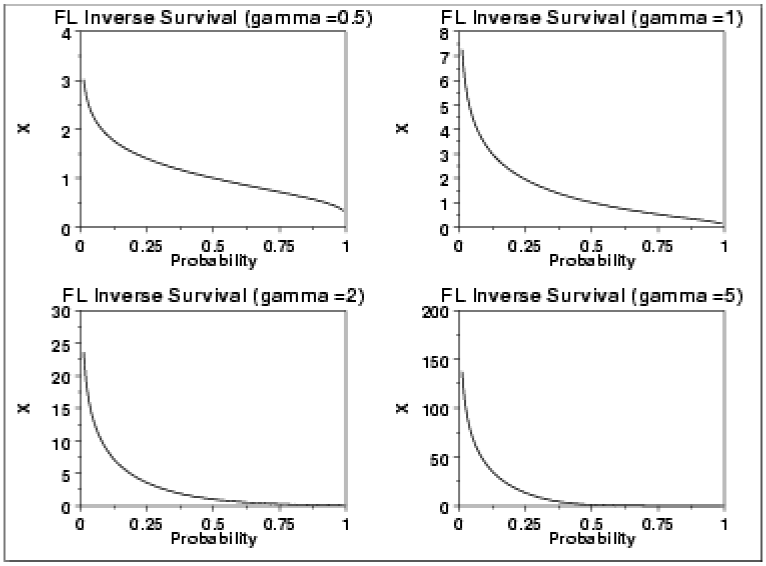
The following is the plot of the Birnbaum-Saunders survival function with the same values of γ as the pdf plots above.



Inverse Survival Function

The Birnbaum-Saunders inverse survival function can be computed from the Birnbaum-Saunders percent point function.

The following is the plot of the gamma inverse survival function with the same values of γ as the pdf plots above.



Common Statistics

The formulas below are with the location parameter equal to zero and the scale parameter equal to one.

Mean	$1 + \frac{\gamma^2}{2}$
Range	Zero to positive infinity.
Standard Deviation	$\gamma\sqrt{1 + \frac{5\gamma^2}{4}}$
Coefficient of Variation	$\frac{2 + \gamma^2}{\gamma\sqrt{1 + 5\gamma^2}}$

Parameter Estimation

Maximum likelihood estimation for the Birnbaum-Saunders distribution is discussed in the Reliability chapter.

Comments

The Birnbaum-Saunders distribution is used extensively in reliability applications to model failure times.

Software

Some general purpose statistical software programs, including Dataplot, support at least some of the probability functions for the Birnbaum-Saunders distribution. Support for this distribution is likely to be available for statistical programs that emphasize reliability applications.

The "bs" package implements support for the Birnbaum-Saunders distribution for the R package. See

> Leiva, V., Hernandez, H., and Riquelme, M. (2006). A New Package for the Birnbaum-Saunders Distribution. *Rnews*, 6/4, 35-40. (http://www.r-project.org)

1.3.6.6.11. Gamma Distribution

Probability Density Function

The general formula for the probability density function of the gamma distribution is

$$f(x) = \frac{\left(\frac{x-\mu}{\beta}\right)^{\gamma-1}\exp\left(-\frac{x-\mu}{\beta}\right)}{\beta\Gamma(\gamma)} \qquad x \ge \mu; \gamma, \beta > 0$$

where γ is the shape parameter, μ is the location parameter, β is the scale parameter, and Γ is the gamma function which has the formula

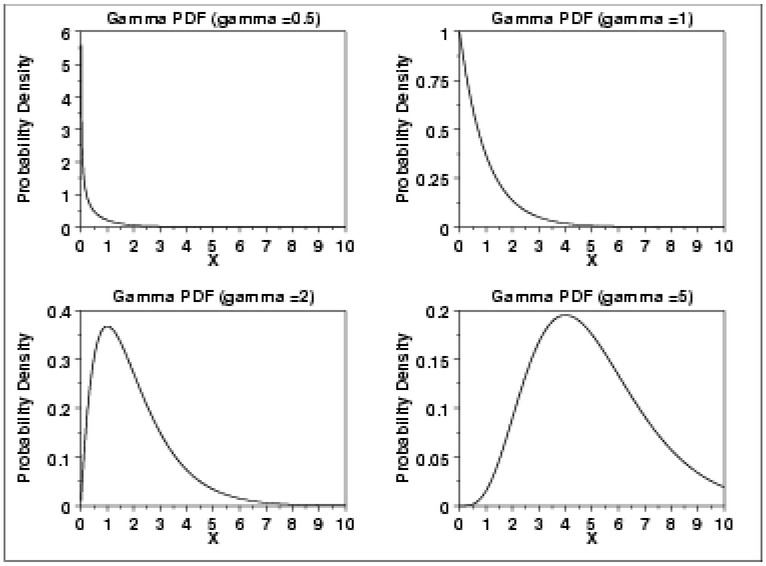
$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$$

The case where μ = 0 and β = 1 is called the **standard gamma distribution**. The equation for the standard gamma distribution reduces to

$$f(x) = \frac{x^{\gamma-1}e^{-x}}{\Gamma(\gamma)} \qquad x \ge 0; \gamma > 0$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the gamma probability density function.



Cumulative Distribution Function

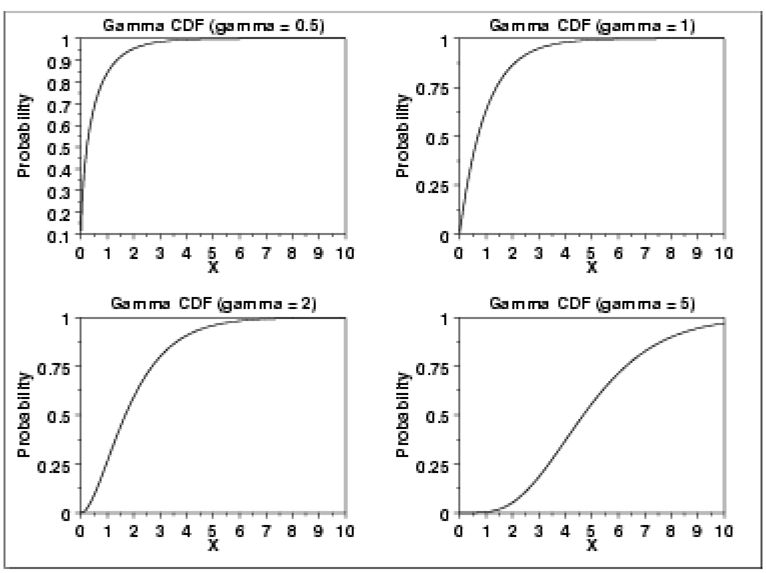
The formula for the cumulative distribution function of the gamma distribution is

$$F(x) = \frac{\Gamma_x(\gamma)}{\Gamma(\gamma)} \qquad x \ge 0; \gamma > 0$$

where Γ is the gamma function defined above and $\Gamma_x(a)$ is the incomplete gamma function. The incomplete gamma function has the formula

$$\Gamma_x(a) = \int_0^x t^{a-1} e^{-t} dt$$

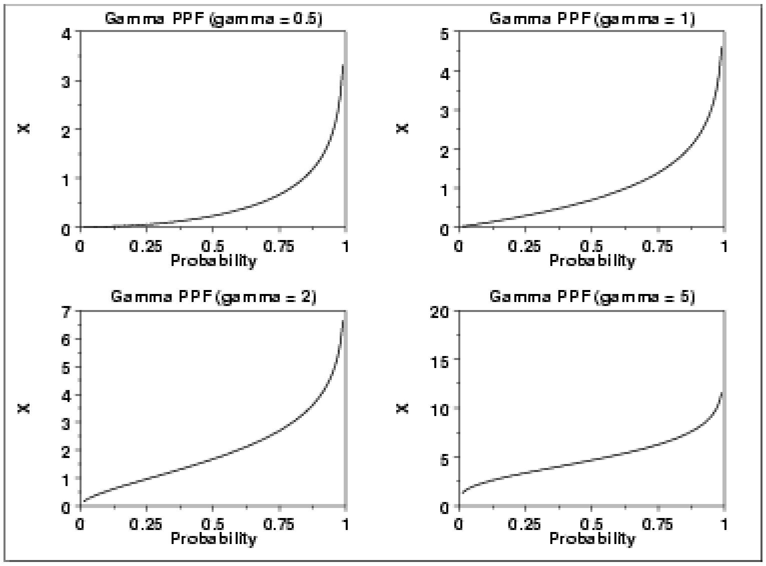
The following is the plot of the gamma cumulative distribution function with the same values of γ as the pdf plots above.



Percent Point Function

The formula for the percent point function of the gamma distribution does not exist in a simple closed form. It is computed numerically.

The following is the plot of the gamma percent point function with the same values of γ as the pdf plots above.

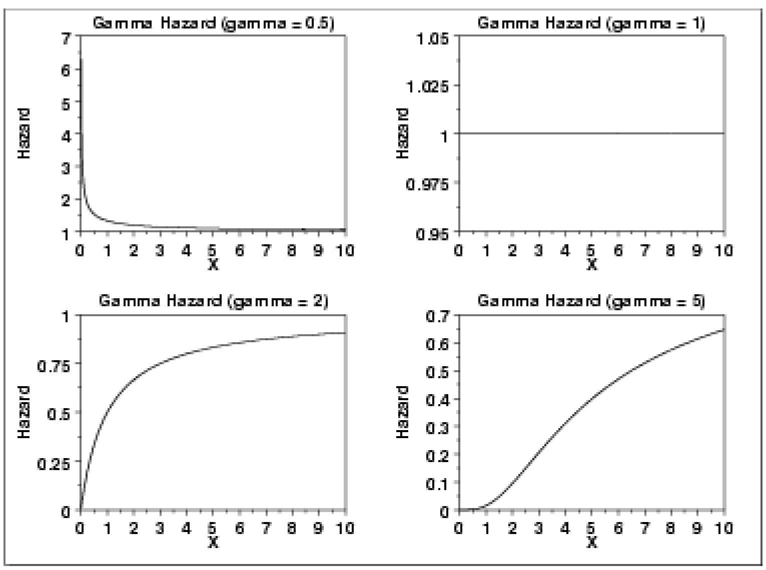


Hazard Function

The formula for the hazard function of the gamma distribution is

$$h(x) = \frac{x^{\gamma-1}e^{-x}}{\Gamma(\gamma) - \Gamma_{x}(\gamma)} \qquad x \ge 0; \gamma > 0$$

The following is the plot of the gamma hazard function with the same values of γ as the pdf plots above.



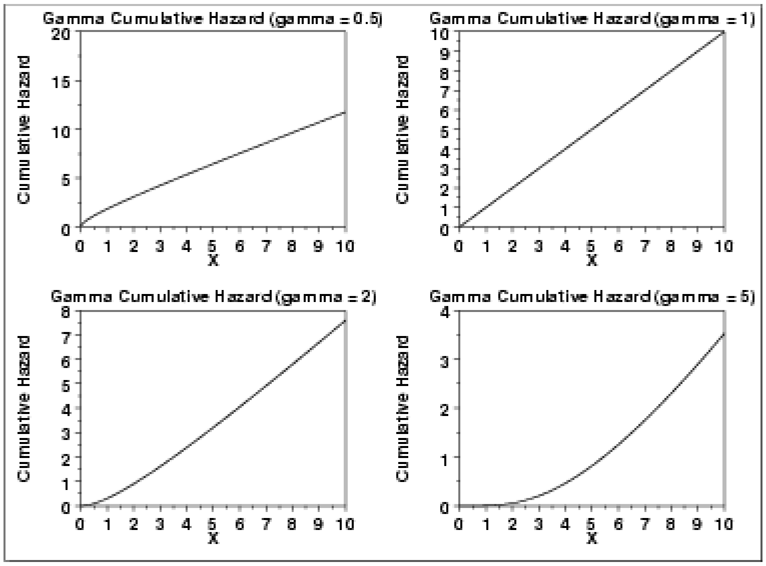
Cumulative Hazard Function

The formula for the cumulative hazard function of the gamma distribution is

$$H(x) = -\log\left(1 - \frac{\Gamma_{x}(\gamma)}{\Gamma(\gamma)}\right) \qquad x \ge 0; \gamma > 0$$

where Γ is the gamma function defined above and $\Gamma_{x}(a)$ is the incomplete gamma function defined above.

The following is the plot of the gamma cumulative hazard function with the same values of γ as the pdf plots above.



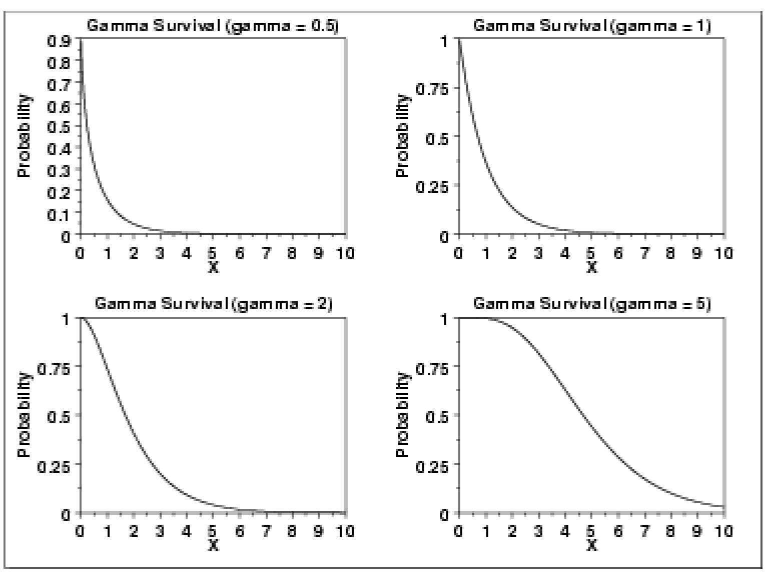
Survival Function

The formula for the survival function of the gamma distribution is

$$S(x) = 1 - \frac{\Gamma_x(\gamma)}{\Gamma(\gamma)} \qquad x \ge 0; \gamma > 0$$

where Γ is the gamma function defined above and $\Gamma_x(a)$ is the incomplete gamma function defined above.

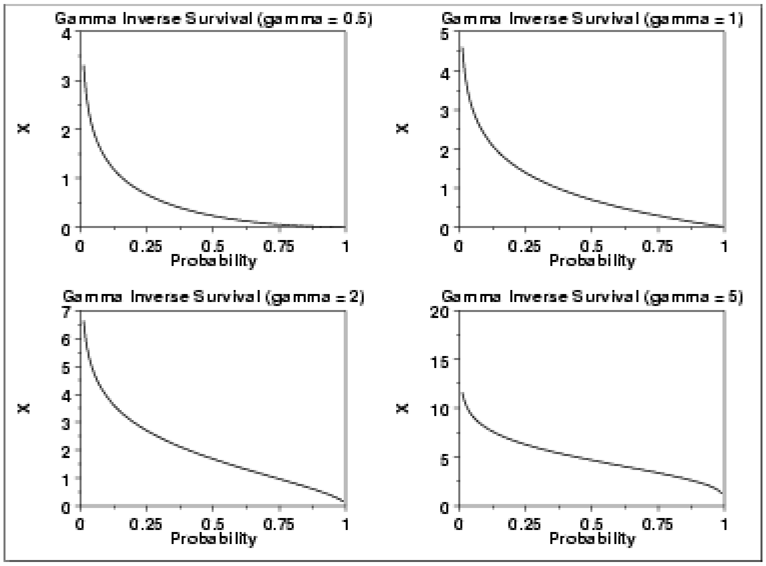
The following is the plot of the gamma survival function with the same values of γ as the pdf plots above.



Inverse Survival Function

The gamma inverse survival function does not exist in simple closed form. It is computed numberically.

The following is the plot of the gamma inverse survival function with the same values of γ as the pdf plots above.



Common Statistics

The formulas below are with the location parameter equal to zero and the scale parameter equal to one.

Mean	γ
Mode	$\gamma - 1 \qquad \gamma \geq 1$
Range	Zero to positive infinity.
Standard Deviation	$\sqrt{\gamma}$
Skewness	$\frac{2}{\sqrt{\gamma}}$
Kurtosis	$3 + \frac{6}{\gamma}$
Coefficient of Variation	$\frac{1}{\sqrt{\gamma}}$

Parameter Estimation

The method of moments estimators of the gamma distribution are

$$\hat{\gamma} = (\frac{\bar{x}}{s})^2$$

$$\hat{\beta} = \frac{s^2}{\bar{x}}$$

where $\bar{x}$ and s are the sample mean and standard deviation, respectively.

The equations for the maximum likelihood estimation of the shape and scale parameters are given in Chapter 18 of Evans, Hastings, and Peacock and Chapter 17 of Johnson, Kotz, and Balakrishnan. These equations need to be solved numerically; this is typically accomplished by using statistical software packages.

Software Some general purpose statistical software programs support at least some of the probability functions for the gamma distribution.

1.3.6.6.12. Double Exponential Distribution

Probability Density Function

The general formula for the probability density function of the double exponential distribution is

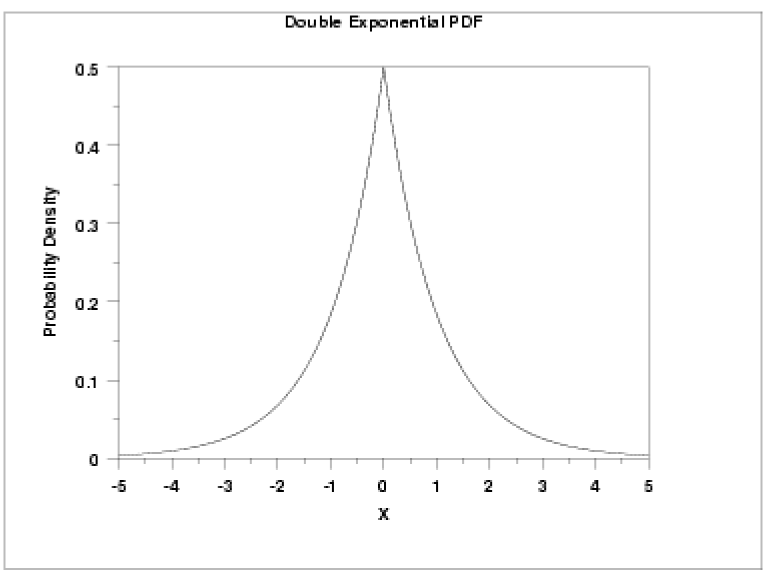
$$f(x) = \frac{e^{-\left|\frac{x-\mu}{\beta}\right|}}{2\beta}$$

where μ is the location parameter and β is the scale parameter. The case where μ = 0 and β = 1 is called the **standard double exponential distribution**. The equation for the standard double exponential distribution is

$$f(x) = \frac{e^{-|x|}}{2}$$

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the double exponential probability density function.



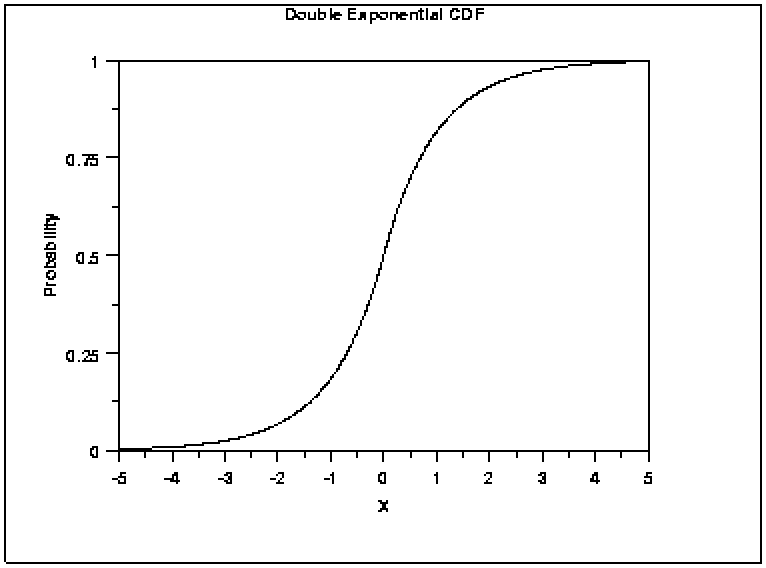
Cumulative Distribution

The formula for the cumulative distribution function of the double exponential distribution is

Function

$$F(x) = \begin{cases} \frac{e^{x}}{2} & \text{for } x < 0 \\ 1 - \frac{e^{-x}}{2} & \text{for } x \ge 0 \end{cases}$$

The following is the plot of the double exponential cumulative distribution function.

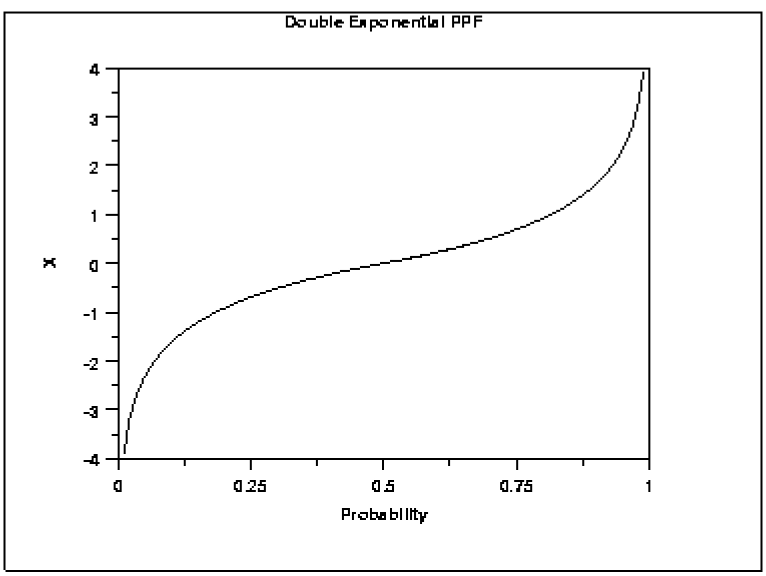


Percent Point Function

The formula for the percent point function of the double exponential distribution is

$$G(P) = \begin{cases} \log(2p) & \text{for } p \le 0.5 \\ -\log(2(1-p)) & \text{for } p > 0.5 \end{cases}$$

The following is the plot of the double exponential percent point function.

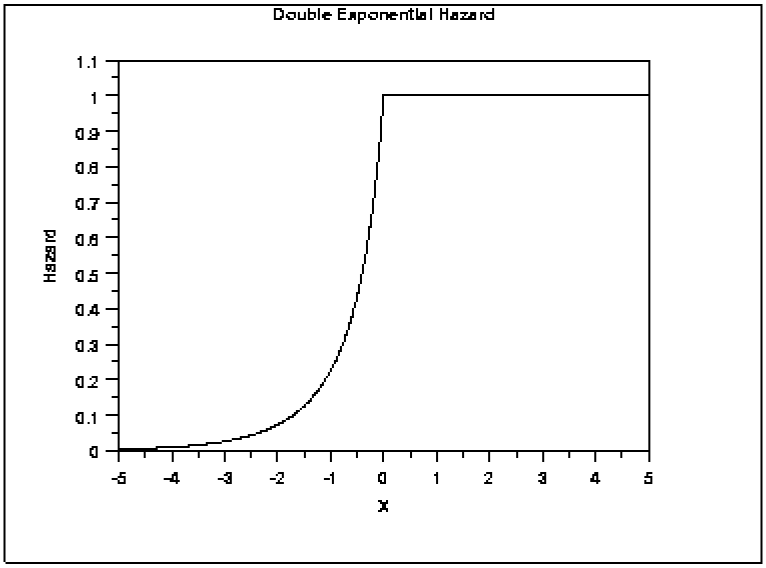


Hazard Function

The formula for the hazard function of the double exponential distribution is

$$h(x) = \begin{cases} \frac{e^{x}}{2-e^{x}} & \text{for } x < 0 \\ 1 & \text{for } x \ge 0 \end{cases}$$

The following is the plot of the double exponential hazard function.

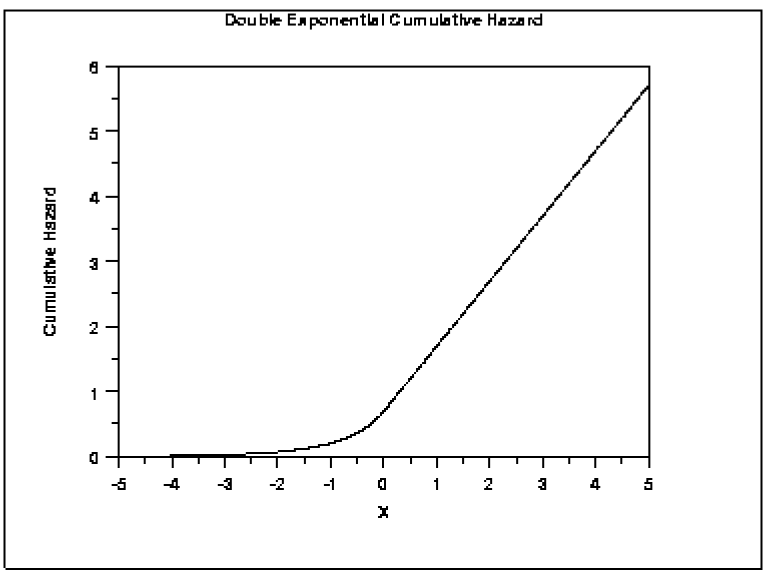


Cumulative Hazard Function

The formula for the cumulative hazard function of the double exponential distribution is

$$H(x) = \begin{cases} -log(1 - \frac{e^{x}}{2}) & \text{for } x < 0 \\ x + \log(2) & \text{for } x \ge 0 \end{cases}$$

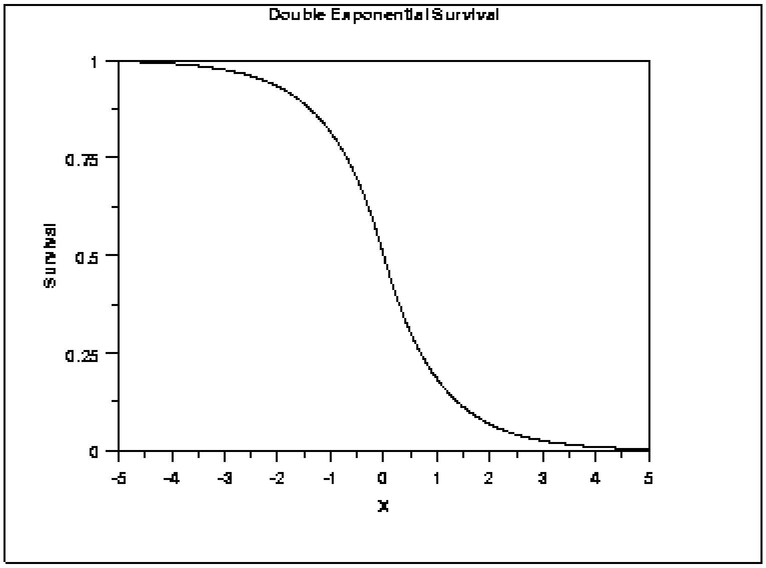
The following is the plot of the double exponential cumulative hazard function.



Survival Function

The double exponential survival function can be computed from the cumulative distribution function of the double exponential distribution.

The following is the plot of the double exponential survival function.

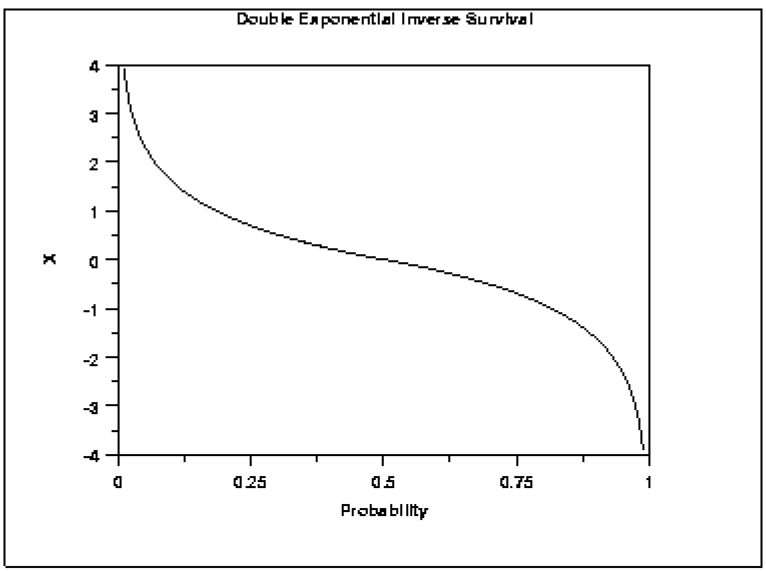


Inverse Survival Function

The formula for the inverse survival function of the double exponential distribution is

$$Z(P) = \begin{cases} \log(2(1-p)) & \text{for } p \le 0.5 \\ -\log(2p) & \text{for } p > 0.5 \end{cases}$$

The following is the plot of the double exponential inverse survival function.



Common Statistics

Mean	μ
Median	μ
Mode	μ
Range	Negative infinity to positive infinity
Standard Deviation	$\sqrt{2}\beta$
Skewness	0
Kurtosis	6

Coefficient of Variation	$\sqrt{2}(\frac{\beta}{\mu})$

Parameter Estimation

The maximum likelihood estimators of the location and scale parameters of the double exponential distribution are

$$\hat{\mu} = \tilde{X}$$

$$\hat{\beta} = \frac{\sum_{i=1}^{N} |X_i - \tilde{X}|}{N}$$

where $\tilde{X}$ is the sample median.

Software

Some general purpose statistical software programs support at least some of the probability functions for the double exponential distribution.

1.3.6.6.13. Power Normal Distribution

Probability Density Function

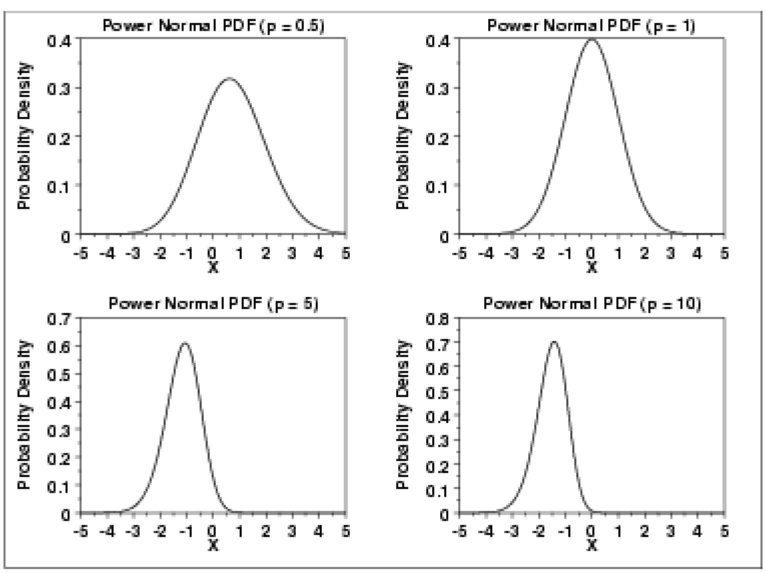
The formula for the probability density function of the standard form of the power normal distribution is

$$f(x,p) = p\phi(x)(\Phi(-x))^{p-1} \qquad x,p>0$$

where p is the shape parameter (also referred to as the power parameter), Φ is the cumulative distribution function of the standard normal distribution, and ϕ is the probability density function of the standard normal distribution.

As with other probability distributions, the power normal distribution can be transformed with a location parameter, μ, and a scale parameter, σ. We omit the equation for the general form of the power normal distribution. Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the power normal probability density function with four values of p.



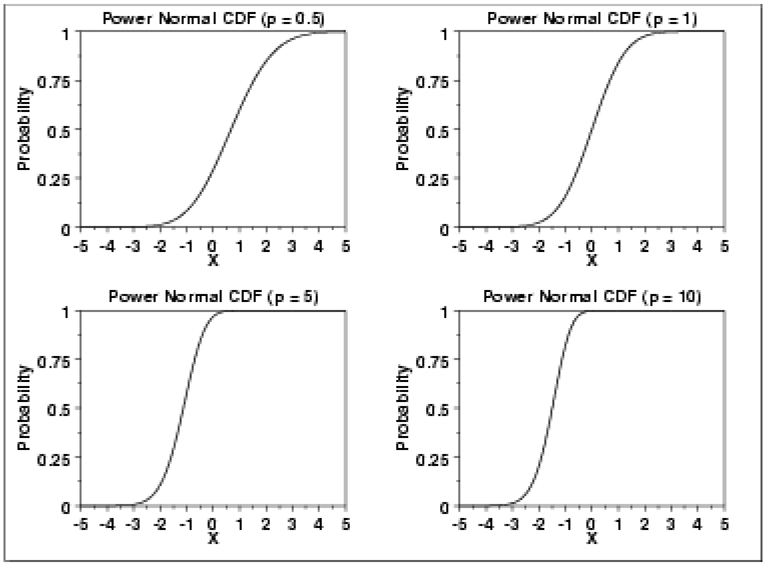
Cumulative Distribution Function

The formula for the cumulative distribution function of the power normal distribution is

$$F(x,p) = 1 - (\Phi(-x))^p \qquad x,p > 0$$

where Φ is the cumulative distribution function of the standard [normal](eda3661.htm) distribution.

The following is the plot of the power normal cumulative distribution function with the same values of p as the pdf plots above.



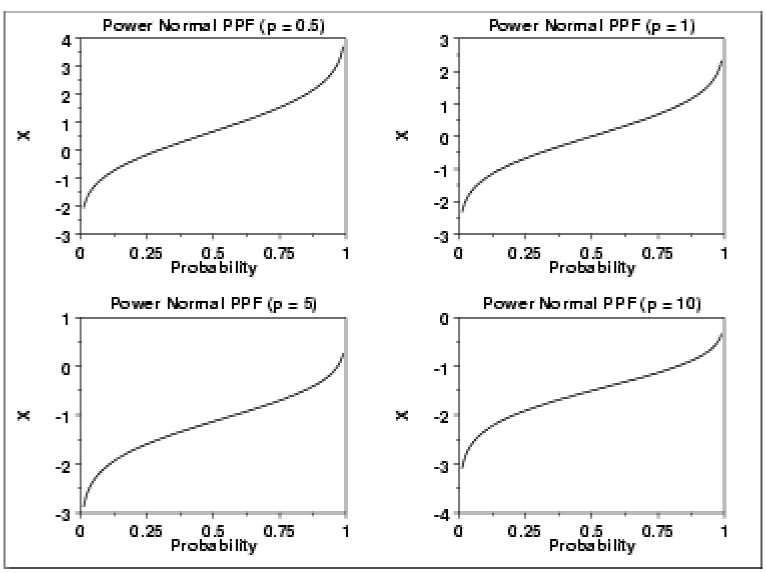
Percent Point Function

The formula for the [percent point function](eda362.htm#PPF) of the power normal distribution is

$$G(f) = \Phi^{-1}(1 - (1-f)^{1/p}) \qquad 0 < f < 1; p > 0$$

where Φ^{-1} is the percent point function of the standard [normal](eda3661.htm) distribution.

The following is the plot of the power normal percent point function with the same values of p as the pdf plots above.



Hazard

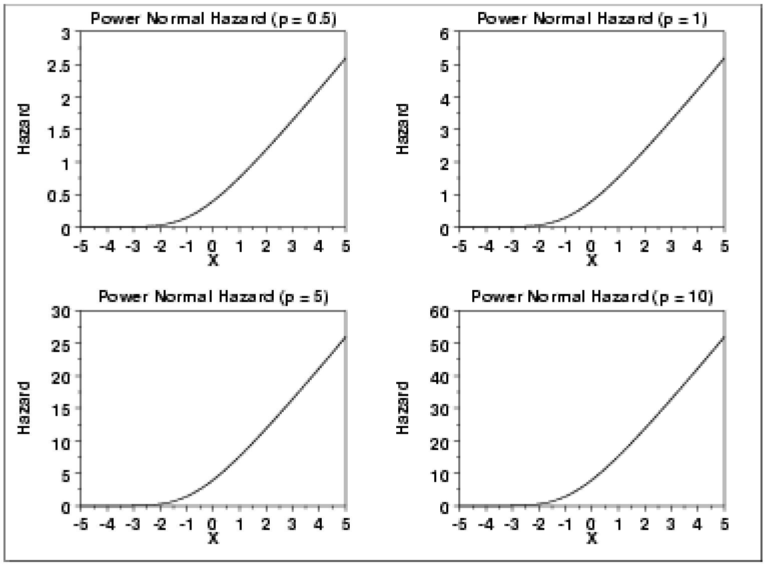
The formula for the [hazard function](eda362.htm#HAZ) of the power normal

Function

distribution is

$$h(x,p) = \frac{p\phi(x)}{\Phi(-x)} \qquad x,p > 0$$

The following is the plot of the power normal hazard function with the same values of ***p*** as the pdf plots above.

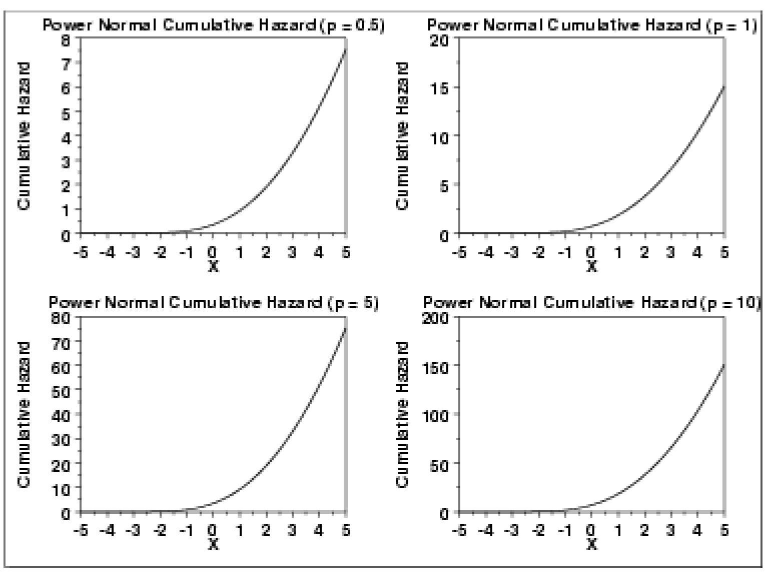


Cumulative Hazard Function

The formula for the cumulative hazard function of the power normal distribution is

$$H(x,p) = -\log\left((\Phi(-x))^{p}\right) \qquad x,p > 0$$

The following is the plot of the power normal cumulative hazard function with the same values of ***p*** as the pdf plots above.

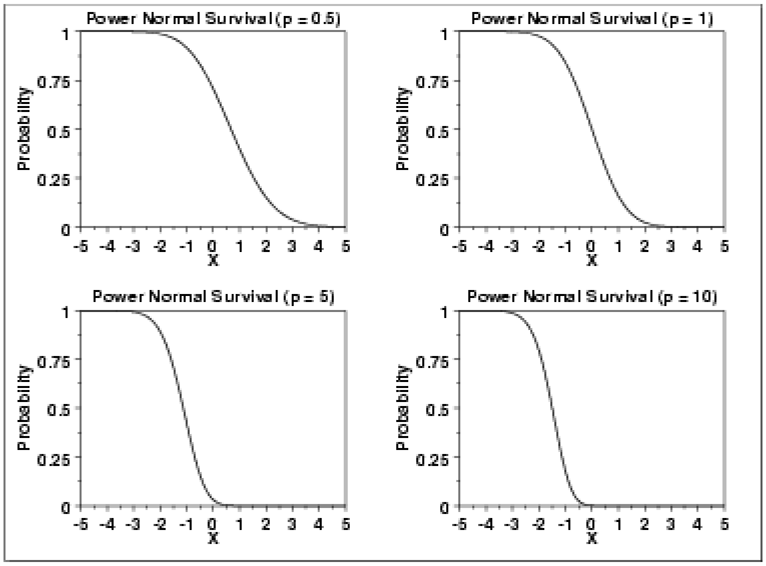


Survival Function

The formula for the survival function of the power normal distribution is

$$S(x,p) = (\Phi(-x))^{p} \qquad x,p > 0$$

The following is the plot of the power normal survival function with the same values of ***p*** as the pdf plots above.

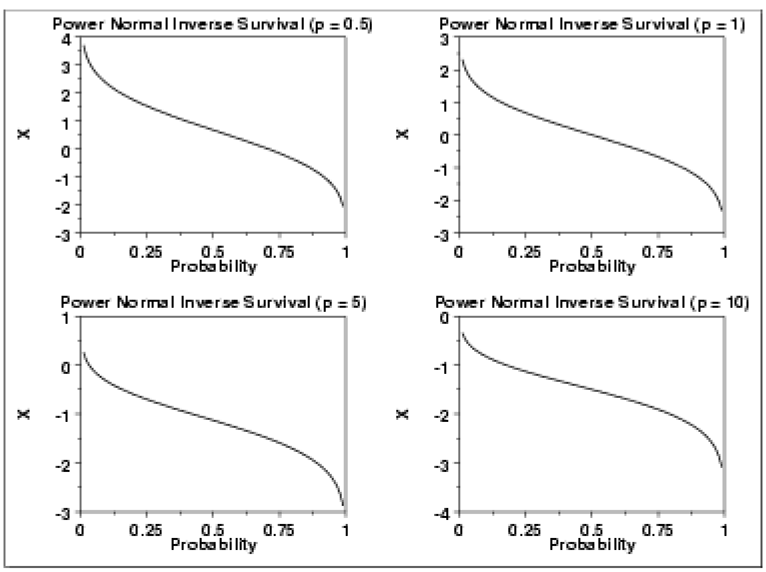


Inverse Survival Function

The formula for the inverse survival function of the power normal distribution is

$$Z(f) = \Phi^{-1}(1 - f^{1/p}) \qquad 0 < f < 1; p > 0$$

The following is the plot of the power normal inverse survival function with the same values of ***p*** as the pdf plots above.



Common Statistics

The statistics for the power normal distribution are complicated and require tables. Nelson discusses the mean, median, mode, and standard deviation of the power normal distribution and provides references to the appropriate tables.

Software

Most general purpose statistical software programs do not support the probability functions for the power normal distribution.

1.3.6.6.14. Power Lognormal Distribution

Probability Density Function

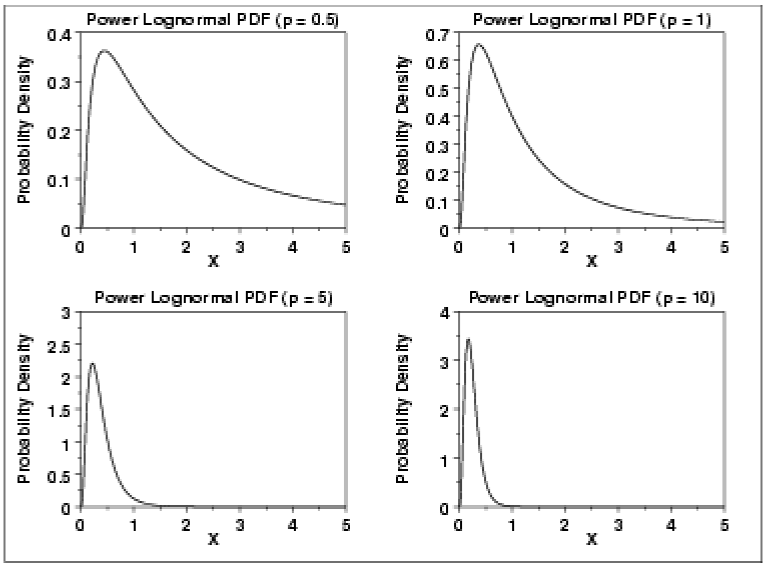
The formula for the probability density function of the standard form of the power lognormal distribution is

$$f(x,p,\sigma) = (\frac{p}{x\sigma})\phi(\frac{\log x}{\sigma})(\Phi(\frac{-\log x}{\sigma}))^{p-1} \qquad x,p,\sigma > 0$$

where p (also referred to as the power parameter) and σ are the shape parameters, Φ is the cumulative distribution function of the standard normal distribution, and ϕ is the probability density function of the standard normal distribution.

As with other probability distributions, the power lognormal distribution can be transformed with a location parameter, μ, and a scale parameter, B. We omit the equation for the general form of the power lognormal distribution. Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the power lognormal probability density function with four values of p and σ set to 1.



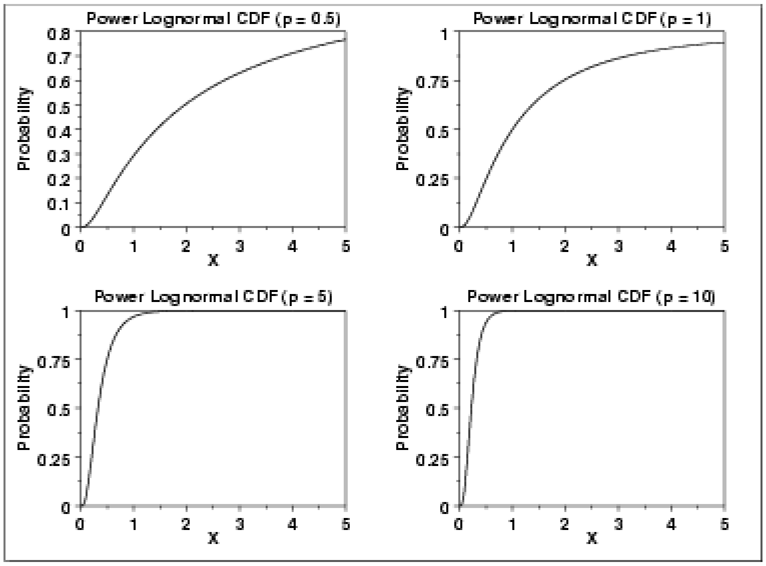
Cumulative Distribution Function

The formula for the cumulative distribution function of the power lognormal distribution is

$$F(x,p,\sigma) = 1 - (\Phi(\frac{-\log x}{\sigma}))^p \qquad x,p,\sigma > 0$$

where Φ is the cumulative distribution function of the standard normal distribution.

The following is the plot of the power lognormal cumulative distribution function with the same values of ***p*** as the pdf plots above.



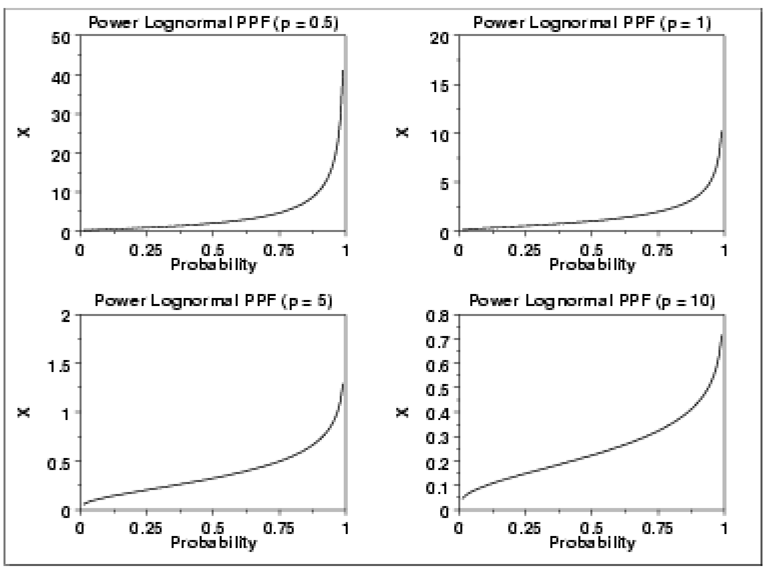
Percent Point Function

The formula for the percent point function of the power lognormal distribution is

$$G(f,p,\sigma) = \exp\left(\Phi^{-1}(1-(1-f)^{1/p})\sigma\right) \qquad 0 < p < 1; p,\sigma > 0$$

where Φ^{-1} is the percent point function of the standard normal distribution.

The following is the plot of the power lognormal percent point function with the same values of ***p*** as the pdf plots above.



Hazard

The formula for the hazard function of the power lognormal distribution

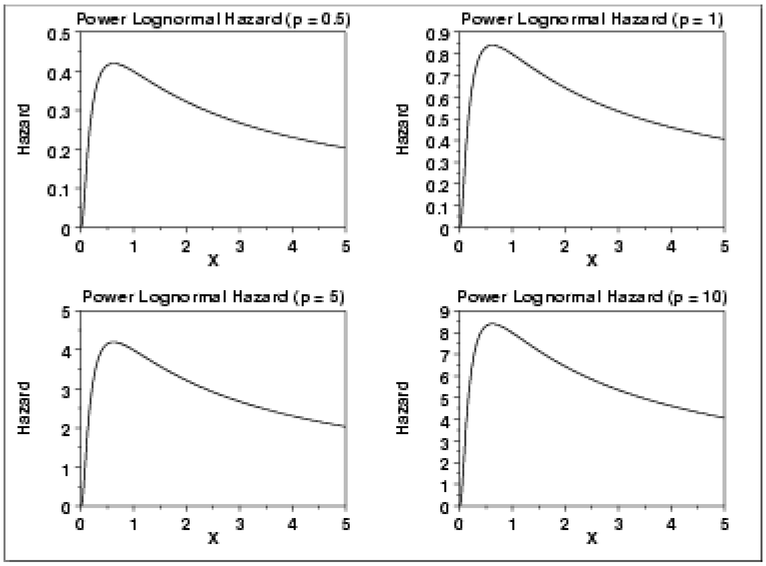
Function is

$$h(x,p,\sigma) = \frac{p(\frac{1}{x\sigma})\phi(\frac{\log x}{\sigma})}{\Phi(\frac{-\log x}{\sigma})} \qquad x,p,\sigma > 0$$

where Φ is the cumulative distribution function of the standard normal distribution, and ϕ is the probability density function of the standard normal distribution.

Note that this is simply a multiple (p) of the lognormal hazard function.

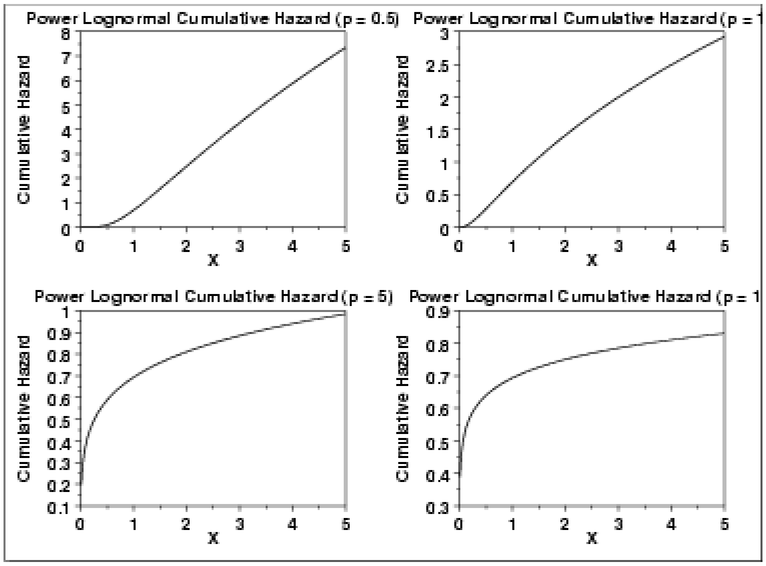
The following is the plot of the power lognormal hazard function with the same values of ***p*** as the pdf plots above.



Cumulative Hazard Function The formula for the cumulative hazard function of the power lognormal distribution is

$$H(x,p,\sigma) = -\log\left((\Phi(\frac{-\log x}{\sigma}))^{p}\right) \qquad x,p,\sigma > 0$$

The following is the plot of the power lognormal cumulative hazard function with the same values of ***p*** as the pdf plots above.

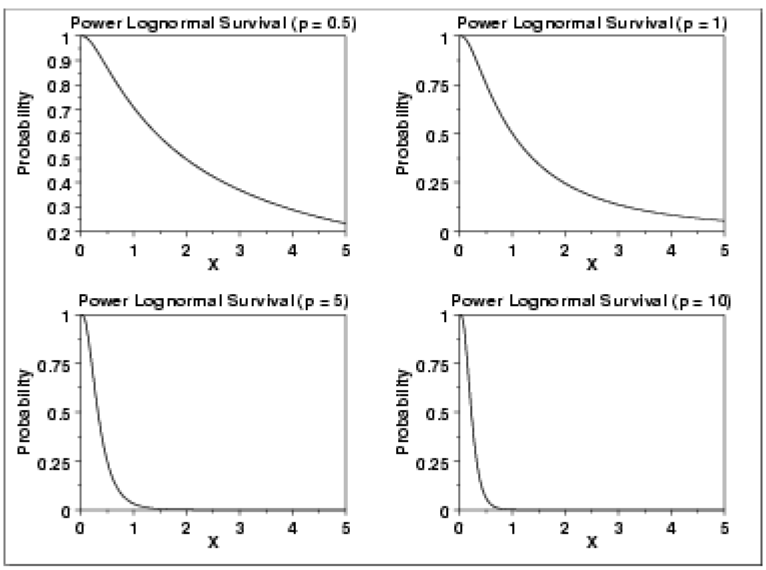


Survival Function

The formula for the survival function of the power lognormal distribution is

$$S(x,p,\sigma) = (\Phi(\frac{-\log x}{\sigma}))^p \qquad x, p, \sigma > 0$$

The following is the plot of the power lognormal survival function with the same values of ***p*** as the pdf plots above.

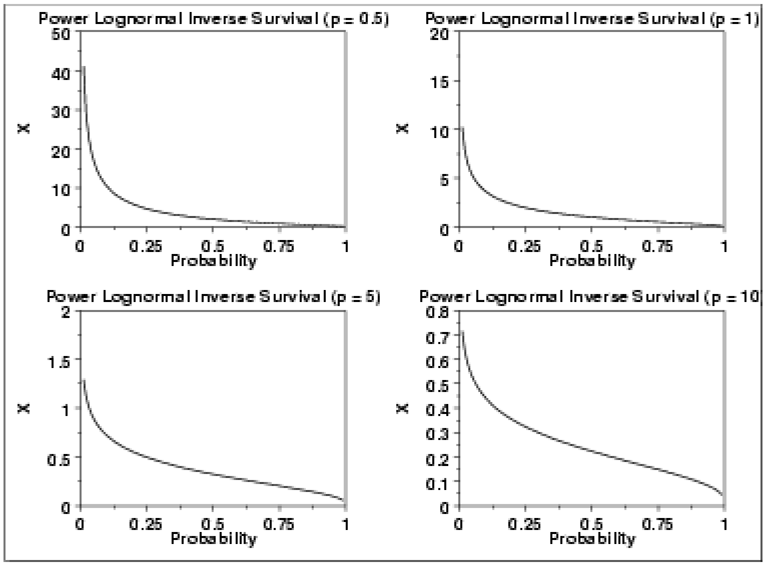


Inverse Survival Function

The formula for the inverse survival function of the power lognormal distribution is

$$Z(f,p,\sigma) = \exp\left(\Phi^{-1}(1 - f^{1/p})\sigma\right) \qquad 0 < p < 1; p, \sigma > 0$$

The following is the plot of the power lognormal inverse survival function with the same values of ***p*** as the pdf plots above.



Common Statistics	The statistics for the power lognormal distribution are complicated and require tables. Nelson discusses the mean, median, mode, and standard deviation of the power lognormal distribution and provides references to the appropriate tables.
Parameter Estimation	Nelson discusses maximum likelihood estimation for the power lognormal distribution. These estimates need to be performed with computer software. Software for maximum likelihood estimation of the parameters of the power lognormal distribution is not as readily available as for other reliability distributions such as the exponential, Weibull, and lognormal.
Software	Most general purpose statistical software programs do not support the probability functions for the power lognormal distribution.

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

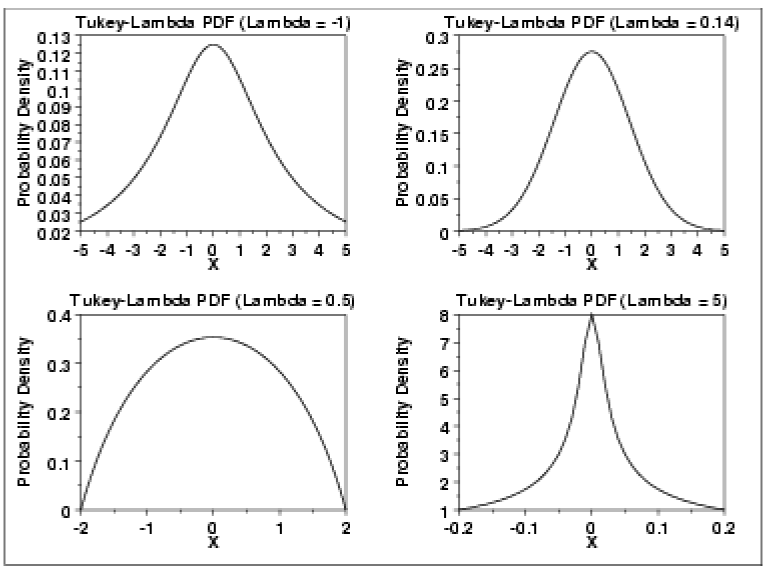
1.3.6.6.15. Tukey-Lambda Distribution

Probability Density Function

The Tukey-Lambda density function does not have a simple, closed form. It is computed numerically.

The Tukey-Lambda distribution has the shape parameter λ. As with other probability distributions, the Tukey-Lambda distribution can be transformed with a location parameter, μ, and a scale parameter, σ. Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

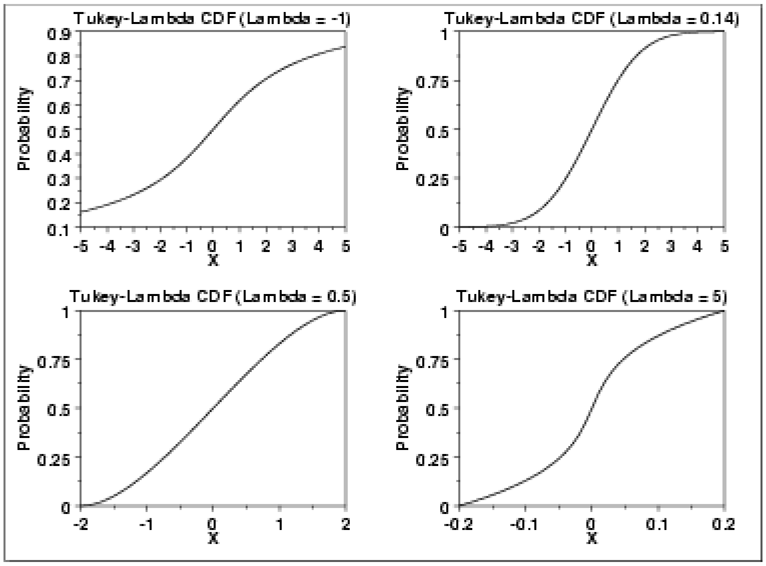
The following is the plot of the Tukey-Lambda probability density function for four values of λ.



Cumulative Distribution Function

The Tukey-Lambda distribution does not have a simple, closed form. It is computed numerically.

The following is the plot of the Tukey-Lambda cumulative distribution function with the same values of λ as the pdf plots above.

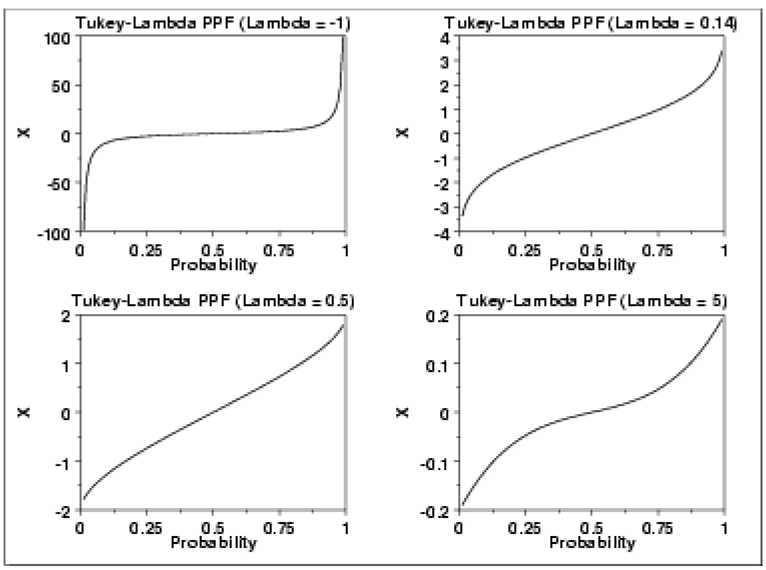


Percent Point Function

The formula for the percent point function of the standard form of the Tukey-Lambda distribution is

$$G(p) = \frac{p^{\lambda} - (1-p)^{\lambda}}{\lambda}$$

The following is the plot of the Tukey-Lambda percent point function with the same values of λ as the pdf plots above.



Other Probability Functions

The Tukey-Lambda distribution is typically used to identify an appropriate distribution (see the comments below) and not used in statistical models directly. For this reason, we omit the formulas, and plots for the hazard, cumulative hazard, survival, and inverse survival functions. We also omit the common statistics and parameter estimation sections.

Comments

The Tukey-Lambda distribution is actually a family of distributions that can approximate a number of common distributions. For example,

λ = -1 approximately Cauchy

λ = 0	exactly logistic
λ = 0.14	approximately normal
λ = 0.5	U-shaped
λ = 1	exactly uniform (from -1 to +1)

The most common use of this distribution is to generate a Tukey-Lambda PPCC plot of a data set. Based on the ppcc plot, an appropriate model for the data is suggested. For example, if the maximum correlation occurs for a value of λ at or near 0.14, then the data can be modeled with a normal distribution. Values of λ less than this imply a heavy-tailed distribution (with -1 approximating a Cauchy). That is, as the optimal value of λ goes from 0.14 to -1, increasingly heavy tails are implied. Similarly, as the optimal value of λ becomes greater than 0.14, shorter tails are implied.

As the Tukey-Lambda distribution is a symmetric distribution, the use of the Tukey-Lambda PPCC plot to determine a reasonable distribution to model the data only applies to symmetric distributuins. A histogram of the data should provide evidence as to whether the data can be reasonably modeled with a symmetric distribution.

Software

Most general purpose statistical software programs do not support the probability functions for the Tukey-Lambda distribution.

1.3.6.6.16. Extreme Value Type I Distribution

Probability Density Function

The extreme value type I distribution has two forms. One is based on the smallest extreme and the other is based on the largest extreme. We call these the minimum and maximum cases, respectively. Formulas and plots for both cases are given. The extreme value type I distribution is also referred to as the Gumbel distribution.

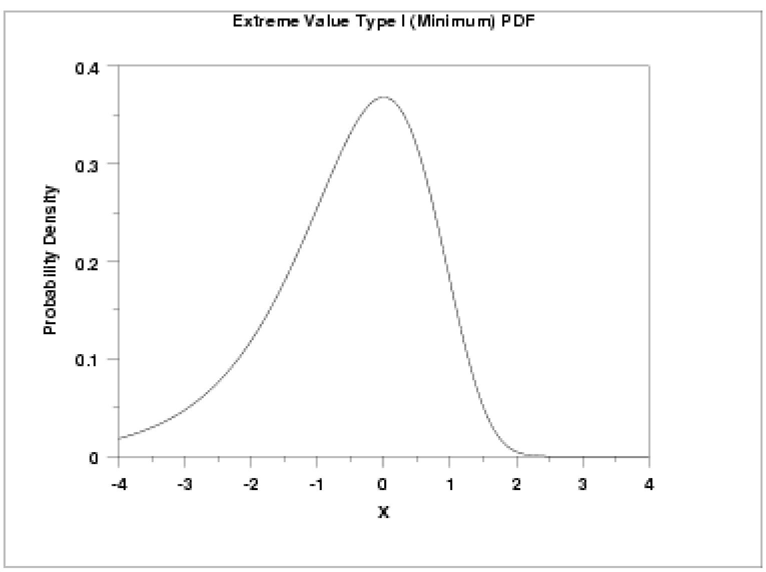
The general formula for the probability density function of the Gumbel (minimum) distribution is

$$f(x) = \frac{1}{\beta} e^{\frac{x-\mu}{\beta}} e^{-e^{\frac{x-\mu}{\beta}}}$$

where μ is the location parameter and β is the scale parameter. The case where μ = 0 and β = 1 is called the **standard Gumbel distribution**. The equation for the standard Gumbel distribution (minimum) reduces to

$$f(x) = e^{x} e^{-e^{x}}$$

The following is the plot of the Gumbel probability density function for the minimum case.



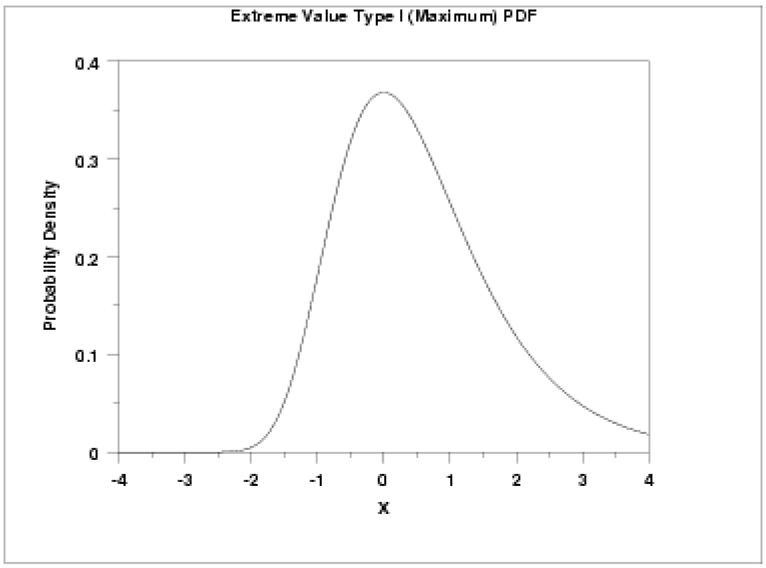
The general formula for the probability density function of the Gumbel (maximum) distribution is

$$f(x) = \frac{1}{\beta} e^{-\frac{x-\mu}{\beta}} e^{-e^{-\frac{x-\mu}{\beta}}}$$

where μ is the location parameter and β is the scale parameter. The case where μ = 0 and β = 1 is called the **standard Gumbel distribution**. The equation for the standard Gumbel distribution (maximum) reduces to

$$f(x) = e^{-x} e^{-e^{-x}}$$

The following is the plot of the Gumbel probability density function for the maximum case.



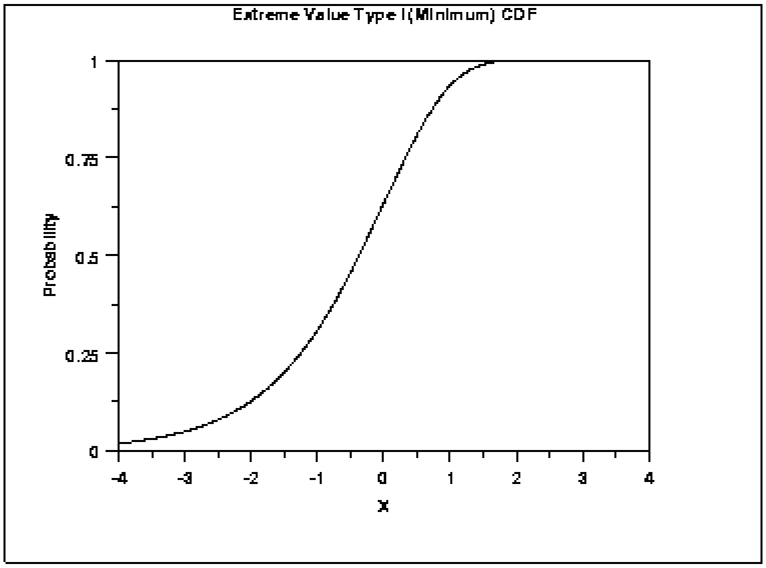
Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

Cumulative Distribution Function

The formula for the cumulative distribution function of the Gumbel distribution (minimum) is

$$F(x) = 1 - e^{-e^{x}}$$

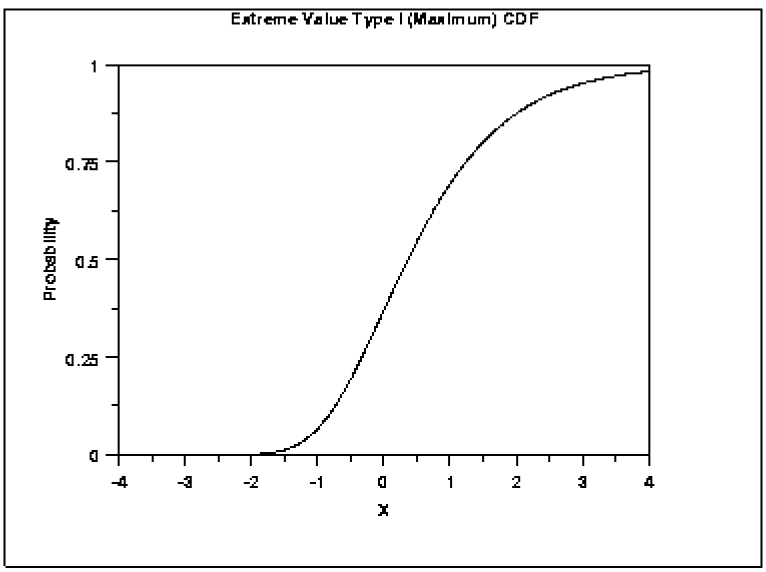
The following is the plot of the Gumbel cumulative distribution function for the minimum case.



The formula for the cumulative distribution function of the Gumbel distribution (maximum) is

$$F(x) = e^{-e^{-x}}$$

The following is the plot of the Gumbel cumulative distribution function for the maximum case.

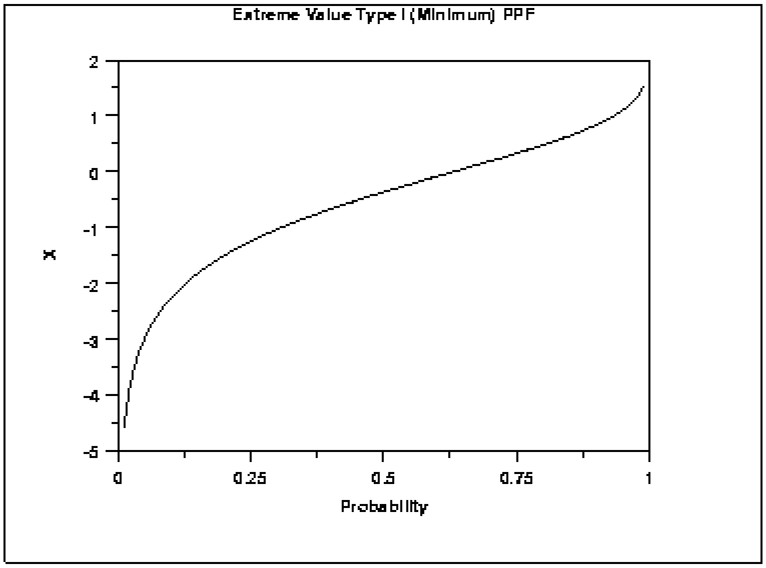


Percent Point Function

The formula for the percent point function of the Gumbel distribution (minimum) is

$$G(p) = \ln(\ln(\frac{1}{1-p}))$$

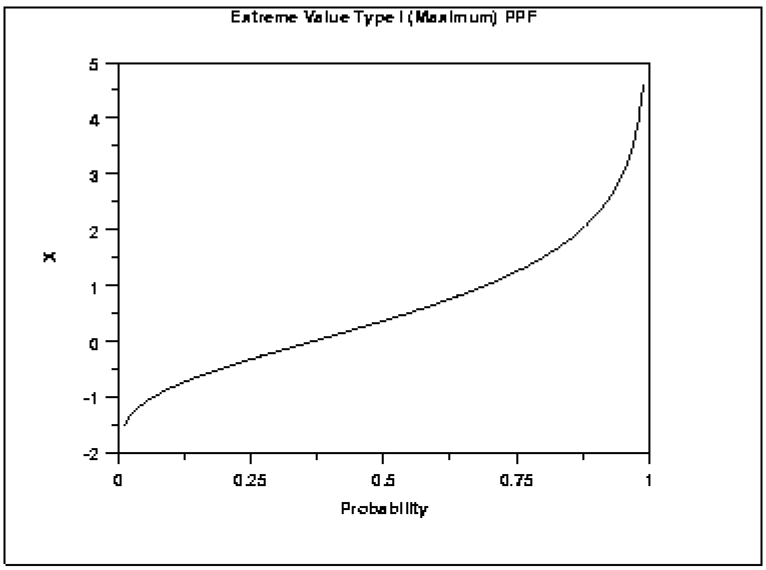
The following is the plot of the Gumbel percent point function for the minimum case.



The formula for the percent point function of the Gumbel distribution (maximum) is

$$G(p) = -\ln(\ln(\frac{1}{p}))$$

The following is the plot of the Gumbel percent point function for the maximum case.

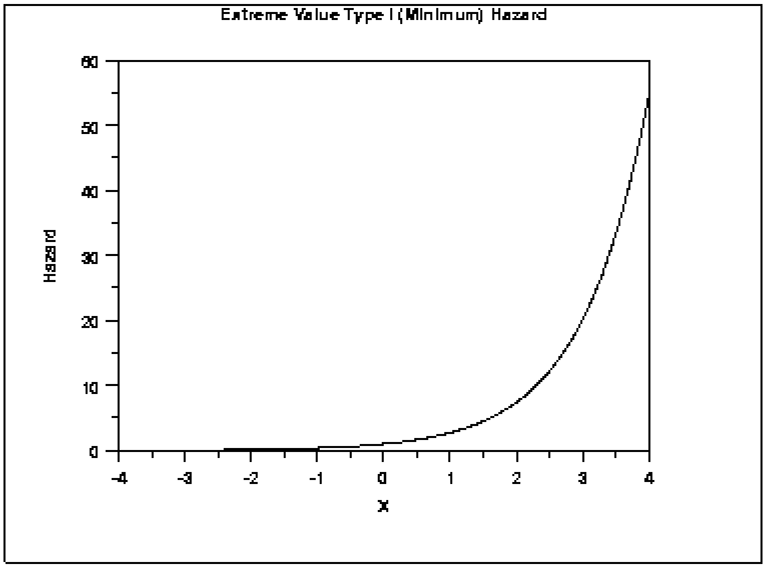


Hazard Function

The formula for the hazard function of the Gumbel distribution (minimum) is

$$h(x) = e^{x}$$

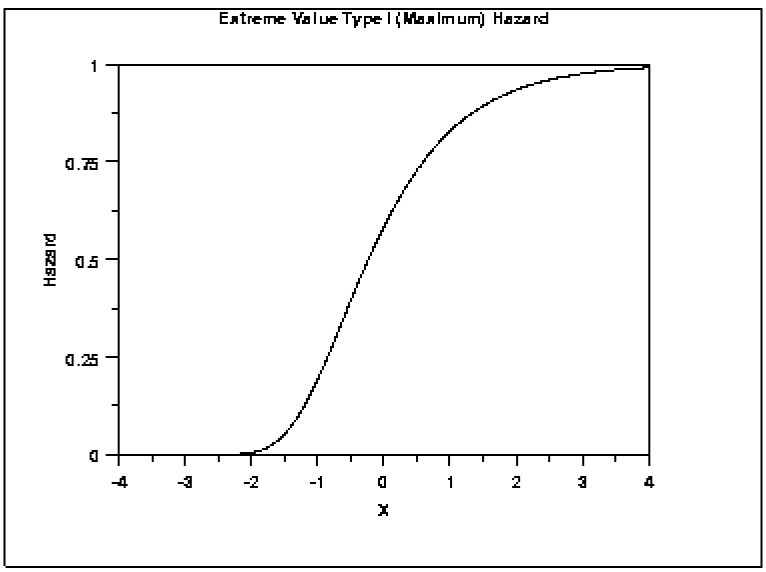
The following is the plot of the Gumbel hazard function for the minimum case.



The formula for the hazard function of the Gumbel distribution (maximum) is

$$h(x) = \frac{e^{-x}}{e^{e^{-x}} - 1}$$

The following is the plot of the Gumbel hazard function for the maximum case.

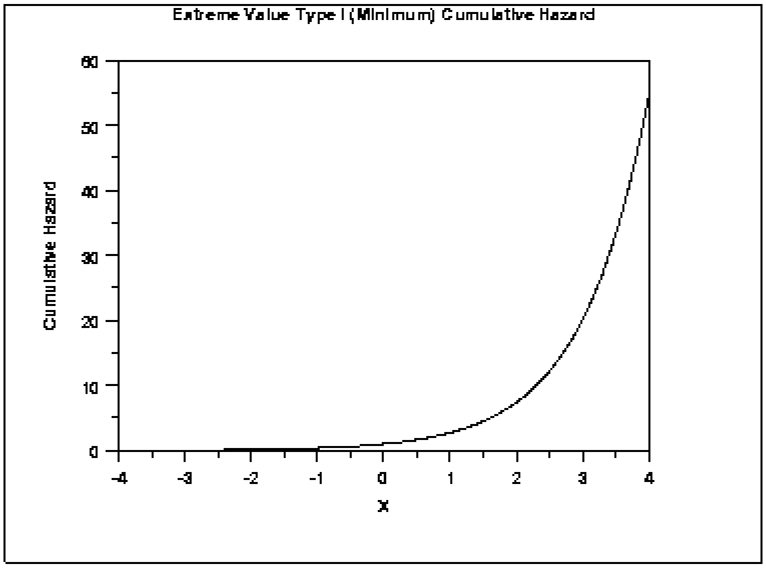


Cumulative Hazard Function

The formula for the cumulative hazard function of the Gumbel distribution (minimum) is

$$H(x) = e^{x}$$

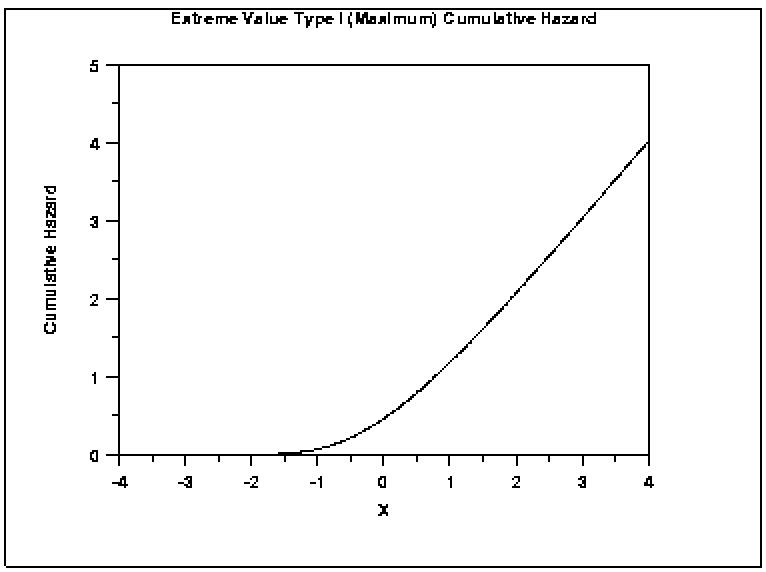
The following is the plot of the Gumbel cumulative hazard function for the minimum case.



The formula for the cumulative hazard function of the Gumbel distribution (maximum) is

$$H(x) = -\ln(1 - e^{-e^{-x}})$$

The following is the plot of the Gumbel cumulative hazard function for the maximum case.

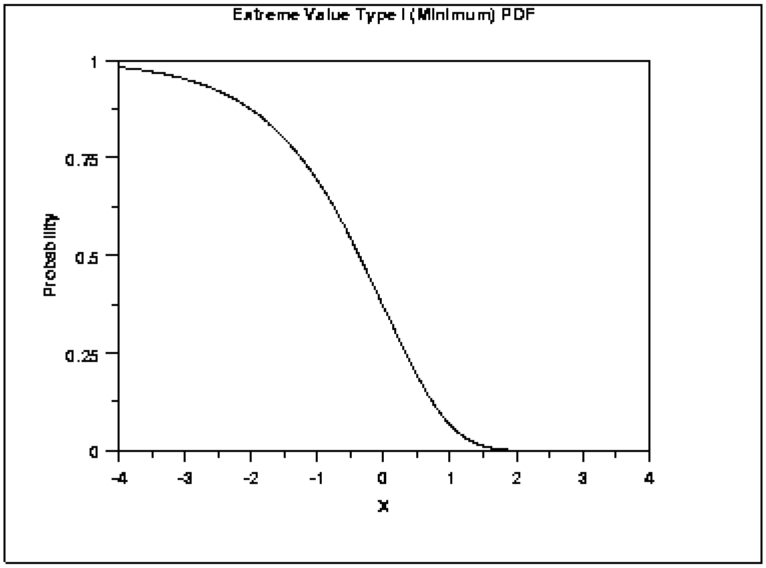


Survival Function

The formula for the survival function of the Gumbel distribution (minimum) is

$$S(x) = e^{-e^{x}}$$

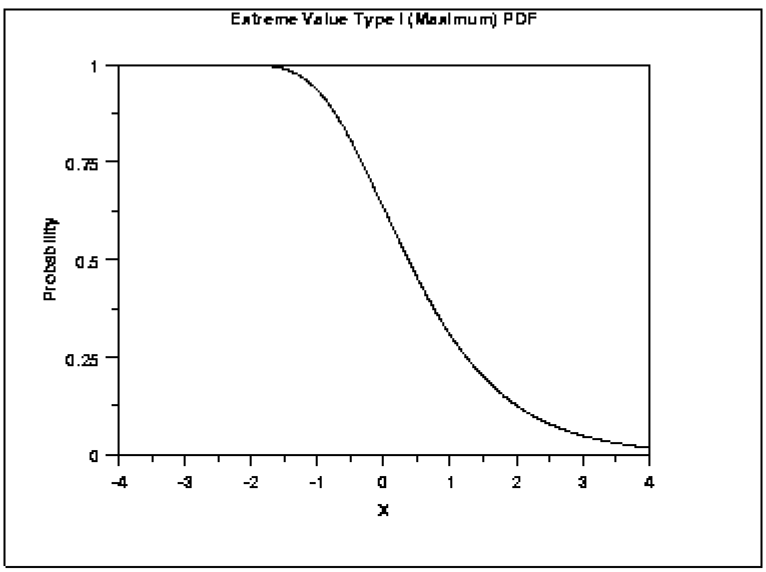
The following is the plot of the Gumbel survival function for the minimum case.



The formula for the survival function of the Gumbel distribution (maximum) is

$$S(x) = 1 - e^{-e^{-x}}$$

The following is the plot of the Gumbel survival function for the maximum case.

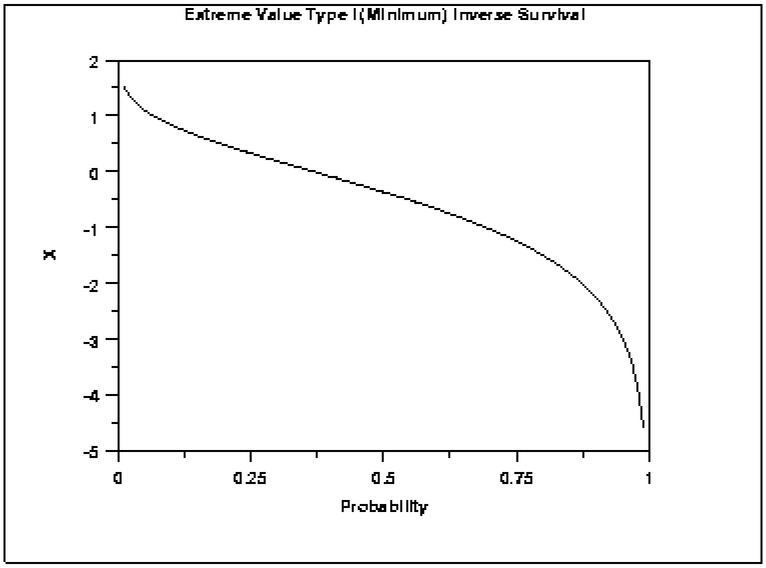


Inverse Survival Function

The formula for the inverse survival function of the Gumbel distribution (minimum) is

$$Z(p) = \ln(\ln(\frac{1}{p}))$$

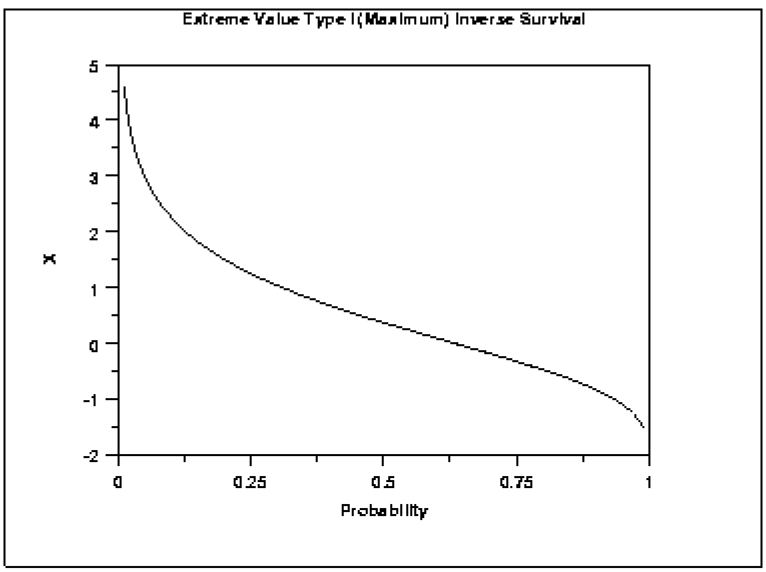
The following is the plot of the Gumbel inverse survival function for the minimum case.



The formula for the inverse survival function of the Gumbel distribution (maximum) is

$$Z(p) = -\ln(\ln(\frac{1}{1-p}))$$

The following is the plot of the Gumbel inverse survival function for the maximum case.



Common Statistics

The formulas below are for the maximum order statistic case.

Mean	$\mu + 0.5772\beta$ The constant 0.5772 is Euler's number.
Median	$\mu - \beta\ln(\ln(2))$
Mode	μ
Range	Negative infinity to positive infinity.
Standard Deviation	$\frac{\beta\pi}{\sqrt{6}}$

Skewness	1.13955
Kurtosis	5.4
Coefficient of Variation	$\frac{\beta\pi}{\sqrt{6}(\mu + 0.5772\beta)}$

Parameter Estimation

The method of moments estimators of the Gumbel (maximum) distribution are

$$\tilde{\beta} = \frac{s\sqrt{6}}{\pi}$$

$$\tilde{\mu} = \bar{X} - 0.5772\tilde{\beta}$$

where $\bar{X}$ and s are the sample mean and standard deviation, respectively.

The equations for the maximum likelihood estimation of the shape and scale parameters are discussed in Chapter 15 of Evans, Hastings, and Peacock and Chapter 22 of Johnson, Kotz, and Balakrishnan. These equations need to be solved numerically and this is typically accomplished by using statistical software packages.

Software

Some general purpose statistical software programs support at least some of the probability functions for the extreme value type I distribution.

1.3.6.6.17. Beta Distribution

Probability Density Function

The general formula for the probability density function of the beta distribution is

$$f(x) = \frac{(x-a)^{p-1}(b-x)^{q-1}}{B(p,q)(b-a)^{p+q-1}} \qquad a \le x \le b; p,q > 0$$

where p and q are the shape parameters, a and b are the lower and upper bounds, respectively, of the distribution, and $B(p,q)$ is the beta function. The beta function has the formula

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

The case where a = 0 and b = 1 is called the **standard beta distribution**. The equation for the standard beta distribution is

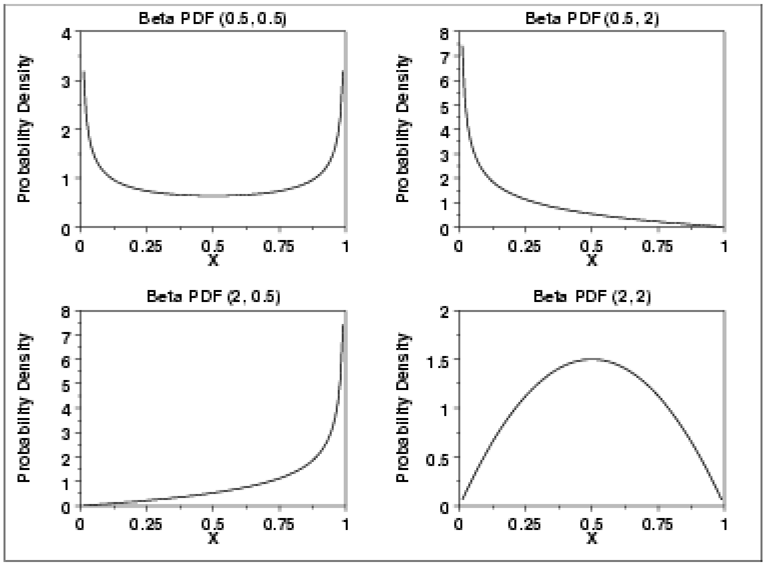
$$f(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)} \qquad 0 \le x \le 1; p,q > 0$$

Typically we define the general form of a distribution in terms of location and scale parameters. The beta is different in that we define the general distribution in terms of the lower and upper bounds. However, the location and scale parameters can be defined in terms of the lower and upper limits as follows:

location = a
scale = b - a

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following is the plot of the beta probability density function for four different values of the shape parameters.



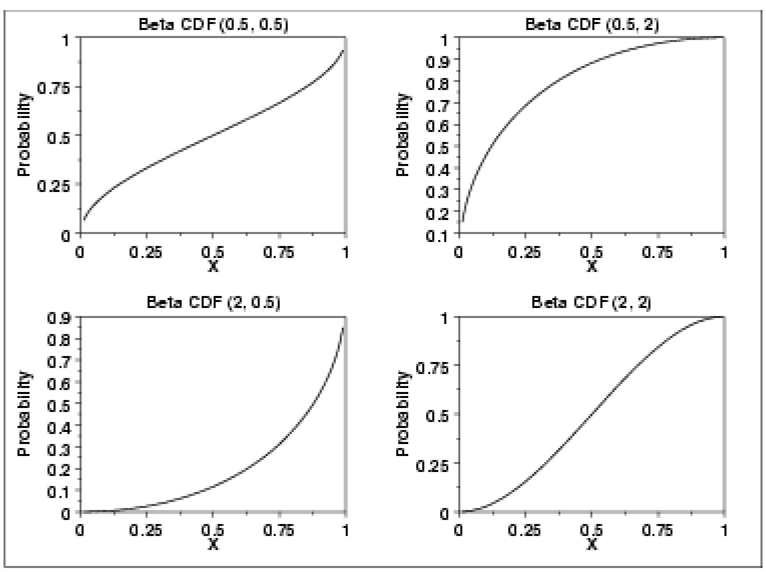
Cumulative Distribution Function

The formula for the cumulative distribution function of the beta distribution is also called the incomplete beta function ratio (commonly denoted by I_x) and is defined as

$$F(x) = I_x(p,q) = \frac{\int_0^x t^{p-1}(1-t)^{q-1}dt}{B(p,q)} \qquad 0 \le x \le 1; p,q > 0$$

where B is the beta function defined above.

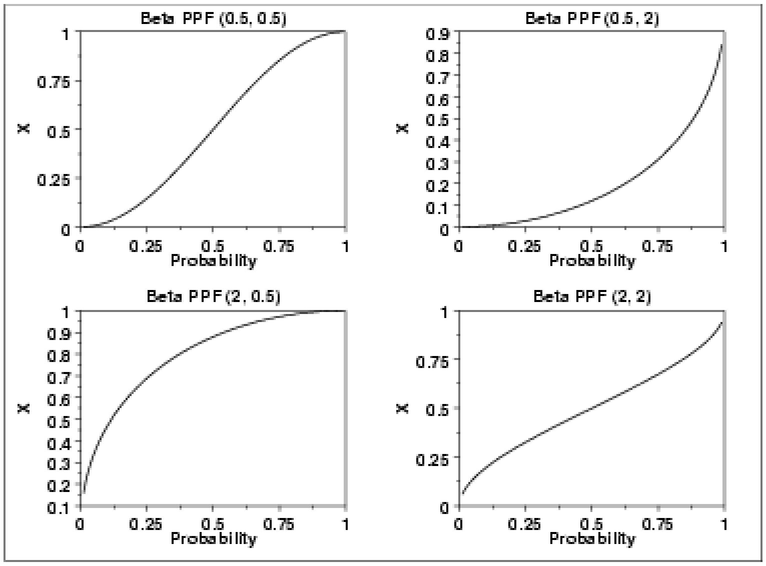
The following is the plot of the beta cumulative distribution function with the same values of the shape parameters as the pdf plots above.



Percent Point Function

The formula for the percent point function of the beta distribution does not exist in a simple closed form. It is computed numerically.

The following is the plot of the beta percent point function with the same values of the shape parameters as the pdf plots above.



Other Probability Functions

Since the beta distribution is not typically used for reliability applications, we omit the formulas and plots for the hazard, cumulative hazard, survival, and inverse survival probability functions.

Common Statistics

The formulas below are for the case where the lower limit is zero and the upper limit is one.

Mean	$\frac{p}{p+q}$
Mode	$\frac{p-1}{p+q-2} \qquad p,q > 1$
Range	0 to 1
Standard Deviation	$\sqrt{\frac{pq}{(P+q)^2(p+q+1)}}$
Coefficient of Variation	$\sqrt{\frac{q}{p(p+q+1)}}$
Skewness	$\frac{2(q-p)\sqrt{p+q+1}}{(p+q+2)\sqrt{pq}}$

Parameter Estimation

First consider the case where *a* and *b* are assumed to be known. For this case, the method of moments estimates are

$$p = \bar{x}(\frac{\bar{x}(1-\bar{x})}{s^2} - 1)$$

$$q = (1-\bar{x})(\frac{\bar{x}(1-\bar{x})}{s^2} - 1)$$

where $\bar{x}$ is the sample mean and s^2 is the sample variance. If *a* and *b* are not 0 and 1, respectively, then replace $\bar{x}$ with $\frac{\bar{x}-a}{b-a}$ and s^2 with $\frac{s^2}{(b-a)^2}$ in the above equations.

For the case when *a* and *b* are known, the maximum likelihood estimates can be obtained by solving the following set of equations

$$\psi(\hat{p}) - \psi(\hat{p} + \hat{q}) = \frac{1}{n}\sum_{i=1}^{n}\log(\frac{Y_i - a}{b - a})$$

$$\psi(\hat{q}) - \psi(\hat{p} + \hat{q}) = \frac{1}{n}\sum_{i=1}^{n}\log(\frac{b - Y_i}{b - a})$$

The maximum likelihood equations for the case when *a* and *b* are not known are given in pages 221-235 of Volume II of Johnson, Kotz, and Balakrishan.

Software Most general purpose statistical software programs support at least some of the probability functions for the beta distribution.

1.3.6.6.18. Binomial Distribution

Probability Mass Function

The binomial distribution is used when there are exactly two mutually exclusive outcomes of a trial. These outcomes are appropriately labeled "success" and "failure". The binomial distribution is used to obtain the probability of observing *x* successes in *N* trials, with the probability of success on a single trial denoted by *p*. The binomial distribution assumes that *p* is fixed for all trials.

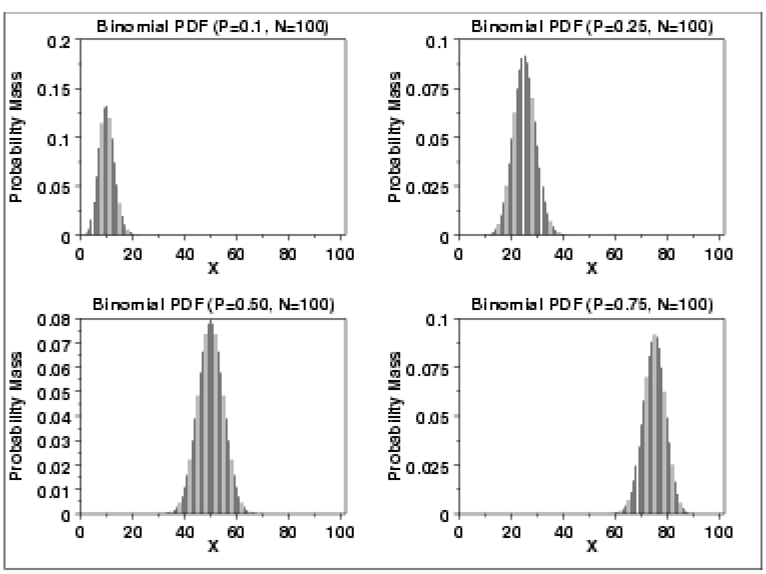
The formula for the binomial probability mass function is

$$P(x,p,n) = \binom{n}{x} (p)^x (1-p)^{(n-x)} \qquad \mbox{for } x = 0, 1, 2, \cdots , n$$

where

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

The following is the plot of the binomial probability density function for four values of *p* and *n* = 100.

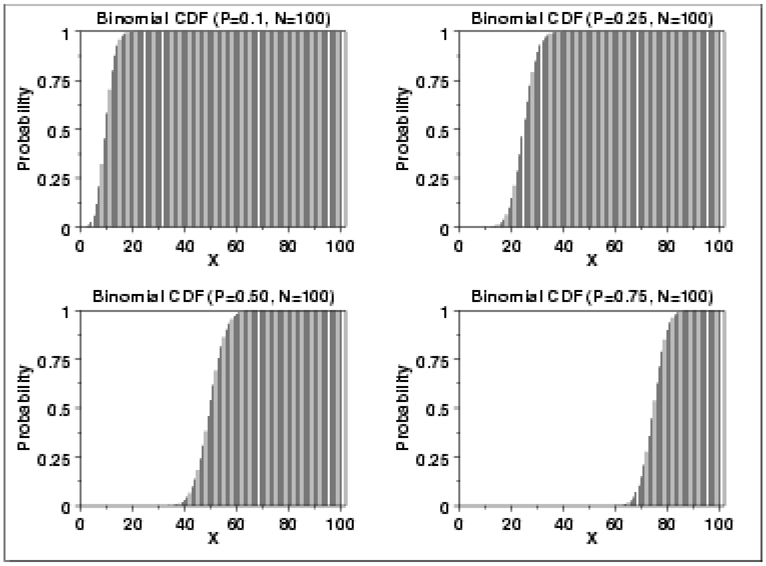


Cumulative Distribution Function

The formula for the binomial cumulative probability function is

$$F(x,p,n) = \sum_{i=0}^{x} \binom{n}{i} (p)^i (1-p)^{(n-i)}$$

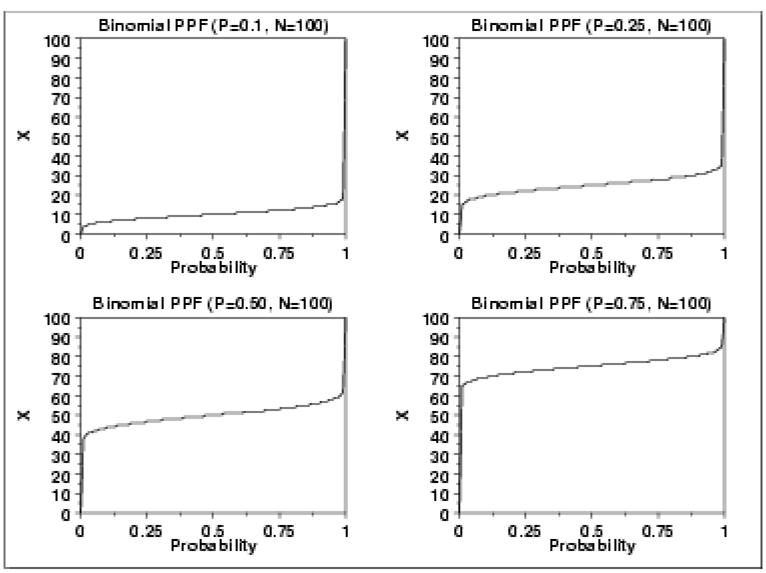
The following is the plot of the binomial cumulative distribution function with the same values of *p* as the pdf plots above.



Percent Point Function

The binomial percent point function does not exist in simple closed form. It is computed numerically. Note that because this is a discrete distribution that is only defined for integer values of *x*, the percent point function is not smooth in the way the percent point function typically is for a continuous distribution.

The following is the plot of the binomial percent point function with the same values of *p* as the pdf plots above.



Common Statistics

Mean	np
Mode	$p(n+1) - 1 \le x \le p(n+1)$
Range	0 to N
Standard Deviation	$\sqrt{np(1-p)}$

Coefficient of Variation	$\sqrt{\frac{(1-p)}{np}}$
Skewness	$\frac{(1-2p)}{\sqrt{np(1-p)}}$
Kurtosis	$3 - \frac{6}{n} + \frac{1}{np(1-p)}$

Comments

The binomial distribution is probably the most commonly used discrete distribution.

Parameter Estimation

The maximum likelihood estimator of *p* (*n* is fixed) is

$$\tilde{p} = \frac{x}{n}$$

Software

Most general purpose statistical software programs support at least some of the probability functions for the binomial distribution.

1.3.6.6.19. Poisson Distribution

Probability Mass Function

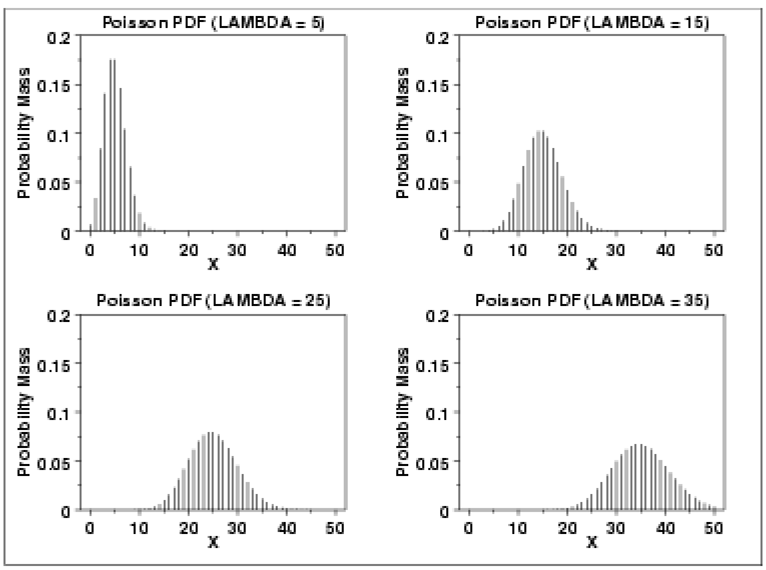
The Poisson distribution is used to model the number of events occurring within a given time interval.

The formula for the Poisson probability mass function is

$$p(x, \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad \mbox{for } x = 0, 1, 2, \cdots$$

λ is the shape parameter which indicates the average number of events in the given time interval.

The following is the plot of the Poisson probability density function for four values of λ.

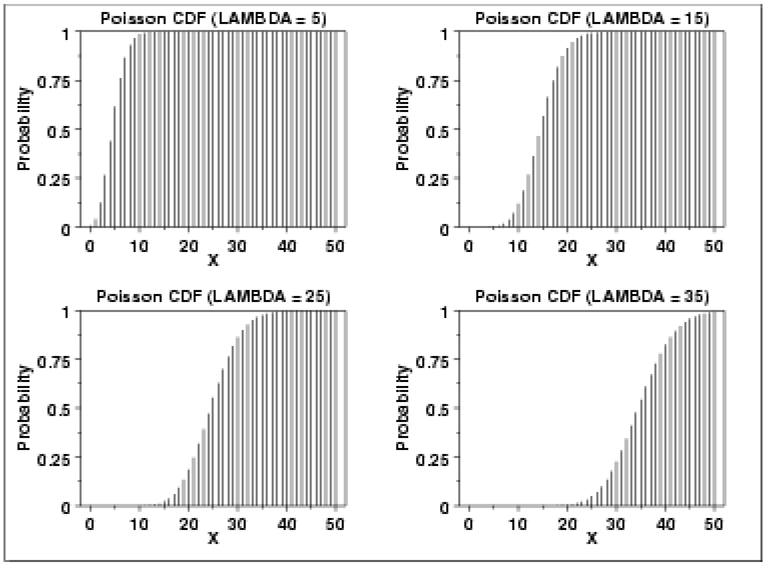


Cumulative Distribution Function

The formula for the Poisson cumulative probability function is

$$F(x, \lambda) = \sum_{i=0}^{x} \frac{e^{-\lambda}\lambda^{i}}{i!}$$

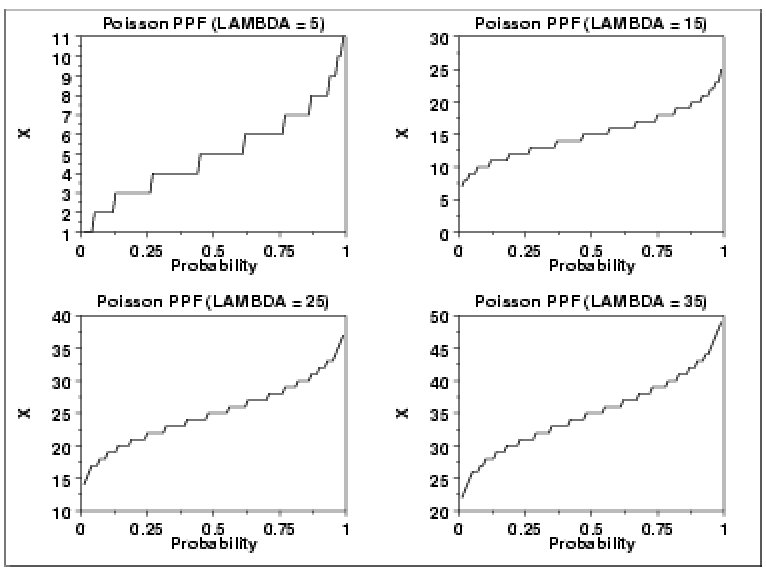
The following is the plot of the Poisson cumulative distribution function with the same values of λ as the pdf plots above.



Percent Point Function

The Poisson percent point function does not exist in simple closed form. It is computed numerically. Note that because this is a discrete distribution that is only defined for integer values of *x*, the percent point function is not smooth in the way the percent point function typically is for a continuous distribution.

The following is the plot of the Poisson percent point function with the same values of λ as the pdf plots above.



Common Statistics

Mean	λ
Mode	For non-integer λ, it is the largest integer less than λ. For integer λ, x = λ and x = λ - 1 are both the mode.
Range	0 to positive infinity
Standard Deviation	$\sqrt{\lambda}$
Coefficient of Variation	$\frac{1}{\sqrt{\lambda}}$

Skewness	$\frac{1}{\sqrt{\lambda}}$
Kurtosis	$3 + \frac{1}{\lambda}$

Parameter Estimation

The maximum likelihood estimator of λ is

$$\tilde{\lambda} = \bar{X}$$

where $\bar{X}$ is the sample mean.

Software

Most general purpose statistical software programs support at least some of the probability functions for the Poisson distribution.

1.3.6.7. Tables for Probability Distributions

Tables

Several commonly used tables for probability distributions can be referenced below.

The values from these tables can also be obtained from most general purpose statistical software programs. Most introductory statistics textbooks (e.g., Snedecor and Cochran) contain more extensive tables than are included here. These tables are included for convenience.

1. Cumulative distribution function for the standard normal distribution
2. Upper critical values of Student's t-distribution with ν degrees of freedom
3. Upper critical values of the F-distribution with ν_1 and ν_2 degrees of freedom
4. Upper critical values of the chi-square distribution with ν degrees of freedom
5. Critical values of t^* distribution for testing the output of a linear calibration line at 3 points
6. Upper critical values of the normal PPCC distribution

1.3.6.7.1. Cumulative Distribution Function of the Standard Normal Distribution

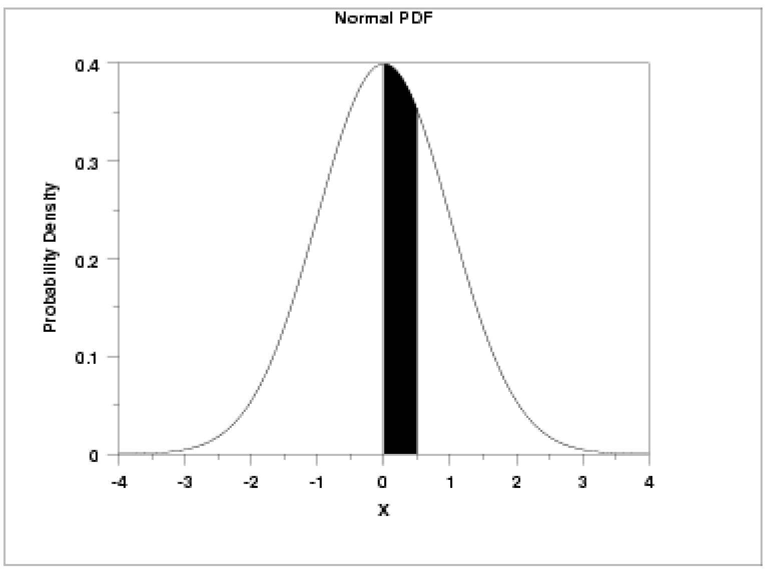
How to Use This Table

The table below contains the area under the standard normal curve from 0 to *z*. This can be used to compute the cumulative distribution function values for the standard normal distribution.

The table utilizes the symmetry of the normal distribution, so what in fact is given is

$$P[0 \le x \le |a|]$$

where *a* is the value of interest. This is demonstrated in the graph below for *a* = 0.5. The shaded area of the curve represents the probability that *x* is between 0 and *a*.



This can be clarified by a few simple examples.

1. What is the probability that *x* is less than or equal to 1.53? Look for 1.5 in the X column, go right to the 0.03 column to find the value 0.43699. Now add 0.5 (for the probability less than zero) to obtain the final result of 0.93699.

2. What is the probability that *x* is less than or equal to -

1.53? For negative values, use the relationship

$$P[x \le a] = 1 - P[x \le |a|] \quad \text{for } x < 0$$

From the first example, this gives 1 - 0.93699 = 0.06301.

3. What is the probability that *x* is between -1 and 0.5? Look up the values for 0.5 (0.5 + 0.19146 = 0.69146) and -1 (1 - (0.5 + 0.34134) = 0.15866). Then subtract the results (0.69146 - 0.15866) to obtain the result 0.5328.

To use this table with a non-standard normal distribution (either the location parameter is not 0 or the scale parameter is not 1), standardize your value by subtracting the mean and dividing the result by the standard deviation. Then look up the value for this standardized value.

A few particularly important numbers derived from the table below, specifically numbers that are commonly used in significance tests, are summarized in the following table:

p	0.001	0.005	0.010	0.025	0.050	0.100
Z_p	-3.090	-2.576	-2.326	-1.960	-1.645	-1.282

p	0.999	0.995	0.990	0.975	0.950	0.900
Z_p	+3.090	+2.576	+2.326	+1.960	+1.645	+1.282

These are critical values for the normal distribution.

Area under the Normal Curve from 0 to X

X	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00000	0.00399	0.00798	0.01197	0.01595	0.01994	0.02392	0.02790	0.03188	0.03586
0.1	0.03983	0.04380	0.04776	0.05172	0.05567	0.05962	0.06356	0.06749	0.07142	0.07535
0.2	0.07926	0.08317	0.08706	0.09095	0.09483	0.09871	0.10257	0.10642	0.11026	0.11409
0.3	0.11791	0.12172	0.12552	0.12930	0.13307	0.13683	0.14058	0.14431	0.14803	0.15173
0.4	0.15542	0.15910	0.16276	0.16640	0.17003	0.17364	0.17724	0.18082	0.18439	0.18793
0.5	0.19146	0.19497	0.19847	0.20194	0.20540	0.20884	0.21226	0.21566	0.21904	0.22240
0.6	0.22575	0.22907	0.23237	0.23565	0.23891	0.24215	0.24537	0.24857	0.25175	0.25490
0.7	0.25804	0.26115	0.26424	0.26730	0.27035	0.27337	0.27637	0.27935	0.28230	0.28524
0.8	0.28814	0.29103	0.29389	0.29673	0.29955	0.30234	0.30511	0.30785	0.31057	0.31327
0.9	0.31594	0.31859	0.32121	0.32381	0.32639	0.32894	0.33147	0.33398	0.33646	0.33891

1.0	0.34134	0.34375	0.34614	0.34849	0.35083	0.35314	0.35543	0.35769	0.35993	0.36214
1.1	0.36433	0.36650	0.36864	0.37076	0.37286	0.37493	0.37698	0.37900	0.38100	0.38298
1.2	0.38493	0.38686	0.38877	0.39065	0.39251	0.39435	0.39617	0.39796	0.39973	0.40147
1.3	0.40320	0.40490	0.40658	0.40824	0.40988	0.41149	0.41308	0.41466	0.41621	0.41774
1.4	0.41924	0.42073	0.42220	0.42364	0.42507	0.42647	0.42785	0.42922	0.43056	0.43189
1.5	0.43319	0.43448	0.43574	0.43699	0.43822	0.43943	0.44062	0.44179	0.44295	0.44408
1.6	0.44520	0.44630	0.44738	0.44845	0.44950	0.45053	0.45154	0.45254	0.45352	0.45449
1.7	0.45543	0.45637	0.45728	0.45818	0.45907	0.45994	0.46080	0.46164	0.46246	0.46327
1.8	0.46407	0.46485	0.46562	0.46638	0.46712	0.46784	0.46856	0.46926	0.46995	0.47062
1.9	0.47128	0.47193	0.47257	0.47320	0.47381	0.47441	0.47500	0.47558	0.47615	0.47670
2.0	0.47725	0.47778	0.47831	0.47882	0.47932	0.47982	0.48030	0.48077	0.48124	0.48169
2.1	0.48214	0.48257	0.48300	0.48341	0.48382	0.48422	0.48461	0.48500	0.48537	0.48574
2.2	0.48610	0.48645	0.48679	0.48713	0.48745	0.48778	0.48809	0.48840	0.48870	0.48899
2.3	0.48928	0.48956	0.48983	0.49010	0.49036	0.49061	0.49086	0.49111	0.49134	0.49158
2.4	0.49180	0.49202	0.49224	0.49245	0.49266	0.49286	0.49305	0.49324	0.49343	0.49361
2.5	0.49379	0.49396	0.49413	0.49430	0.49446	0.49461	0.49477	0.49492	0.49506	0.49520
2.6	0.49534	0.49547	0.49560	0.49573	0.49585	0.49598	0.49609	0.49621	0.49632	0.49643
2.7	0.49653	0.49664	0.49674	0.49683	0.49693	0.49702	0.49711	0.49720	0.49728	0.49736
2.8	0.49744	0.49752	0.49760	0.49767	0.49774	0.49781	0.49788	0.49795	0.49801	0.49807
2.9	0.49813	0.49819	0.49825	0.49831	0.49836	0.49841	0.49846	0.49851	0.49856	0.49861
3.0	0.49865	0.49869	0.49874	0.49878	0.49882	0.49886	0.49889	0.49893	0.49896	0.49900
3.1	0.49903	0.49906	0.49910	0.49913	0.49916	0.49918	0.49921	0.49924	0.49926	0.49929
3.2	0.49931	0.49934	0.49936	0.49938	0.49940	0.49942	0.49944	0.49946	0.49948	0.49950
3.3	0.49952	0.49953	0.49955	0.49957	0.49958	0.49960	0.49961	0.49962	0.49964	0.49965
3.4	0.49966	0.49968	0.49969	0.49970	0.49971	0.49972	0.49973	0.49974	0.49975	0.49976
3.5	0.49977	0.49978	0.49978	0.49979	0.49980	0.49981	0.49981	0.49982	0.49983	0.49983
3.6	0.49984	0.49985	0.49985	0.49986	0.49986	0.49987	0.49987	0.49988	0.49988	0.49989
3.7	0.49989	0.49990	0.49990	0.49990	0.49991	0.49991	0.49992	0.49992	0.49992	0.49992
3.8	0.49993	0.49993	0.49993	0.49994	0.49994	0.49994	0.49994	0.49995	0.49995	0.49995
3.9	0.49995	0.49995	0.49996	0.49996	0.49996	0.49996	0.49996	0.49996	0.49997	0.49997
4.0	0.49997	0.49997	0.49997	0.49997	0.49997	0.49997	0.49998	0.49998	0.49998	0.49998

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.6. Probability Distributions
1.3.6.7. Tables for Probability Distributions

1.3.6.7.2. Critical Values of the Student's *t* Distribution

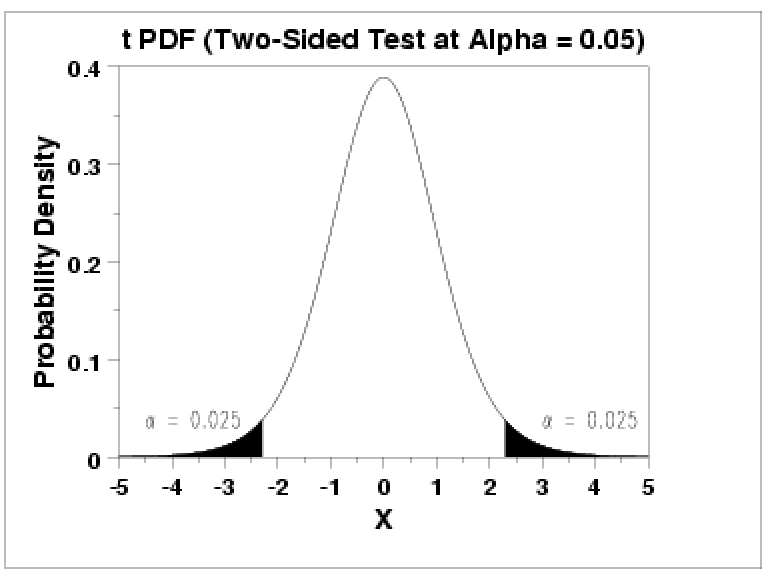
How to Use This Table

This table contains critical values of the Student's *t* distribution computed using the cumulative distribution function. The *t* distribution is symmetric so that

$$t_{1-\alpha,\nu} = -t_{\alpha,\nu}.$$

The *t* table can be used for both one-sided (lower and upper) and two-sided tests using the appropriate value of α.

The significance level, α, is demonstrated in the graph below, which displays a *t* distribution with 10 degrees of freedom. The most commonly used significance level is $\alpha = 0.05$. For a two-sided test, we compute $1 - \alpha/2$, or $1 - 0.05/2 = 0.975$ when $\alpha = 0.05$. If the absolute value of the test statistic is greater than the critical value (0.975), then we reject the null hypothesis. Due to the symmetry of the *t* distribution, we only tabulate the positive critical values in the table below.



Given a specified value for α :

1. For a two-sided test, find the column corresponding to $1-\alpha/2$ and reject the null hypothesis if the absolute value

of the test statistic is greater than the value of $t_{1-\alpha/2,\nu}$ in the table below.

2. For an upper, one-sided test, find the column corresponding to 1-α and reject the null hypothesis if the test statistic is greater than the table value.
3. For a lower, one-sided test, find the column corresponding to 1-α and reject the null hypothesis if the test statistic is less than the negative of the table value.

Critical values of Student's *t* distribution with *ν* degrees of freedom

Probability less than the critical value ($t_{1-\alpha,\nu}$)

ν	0.90	0.95	0.975	0.99	0.995	0.999
1.	3.078	6.314	12.706	31.821	63.657	318.313
2.	1.886	2.920	4.303	6.965	9.925	22.327
3.	1.638	2.353	3.182	4.541	5.841	10.215
4.	1.533	2.132	2.776	3.747	4.604	7.173
5.	1.476	2.015	2.571	3.365	4.032	5.893
6.	1.440	1.943	2.447	3.143	3.707	5.208
7.	1.415	1.895	2.365	2.998	3.499	4.782
8.	1.397	1.860	2.306	2.896	3.355	4.499
9.	1.383	1.833	2.262	2.821	3.250	4.296
10.	1.372	1.812	2.228	2.764	3.169	4.143
11.	1.363	1.796	2.201	2.718	3.106	4.024
12.	1.356	1.782	2.179	2.681	3.055	3.929
13.	1.350	1.771	2.160	2.650	3.012	3.852
14.	1.345	1.761	2.145	2.624	2.977	3.787
15.	1.341	1.753	2.131	2.602	2.947	3.733
16.	1.337	1.746	2.120	2.583	2.921	3.686
17.	1.333	1.740	2.110	2.567	2.898	3.646
18.	1.330	1.734	2.101	2.552	2.878	3.610
19.	1.328	1.729	2.093	2.539	2.861	3.579
20.	1.325	1.725	2.086	2.528	2.845	

						3.552
21.	1.323	1.721	2.080	2.518	2.831	3.527
22.	1.321	1.717	2.074	2.508	2.819	3.505
23.	1.319	1.714	2.069	2.500	2.807	3.485
24.	1.318	1.711	2.064	2.492	2.797	3.467
25.	1.316	1.708	2.060	2.485	2.787	3.450
26.	1.315	1.706	2.056	2.479	2.779	3.435
27.	1.314	1.703	2.052	2.473	2.771	3.421
28.	1.313	1.701	2.048	2.467	2.763	3.408
29.	1.311	1.699	2.045	2.462	2.756	3.396
30.	1.310	1.697	2.042	2.457	2.750	3.385
31.	1.309	1.696	2.040	2.453	2.744	3.375
32.	1.309	1.694	2.037	2.449	2.738	3.365
33.	1.308	1.692	2.035	2.445	2.733	3.356
34.	1.307	1.691	2.032	2.441	2.728	3.348
35.	1.306	1.690	2.030	2.438	2.724	3.340
36.	1.306	1.688	2.028	2.434	2.719	3.333
37.	1.305	1.687	2.026	2.431	2.715	3.326
38.	1.304	1.686	2.024	2.429	2.712	3.319
39.	1.304	1.685	2.023	2.426	2.708	3.313
40.	1.303	1.684	2.021	2.423	2.704	3.307
41.	1.303	1.683	2.020	2.421	2.701	3.301
42.	1.302	1.682	2.018	2.418	2.698	3.296
43.	1.302	1.681	2.017	2.416	2.695	3.291
44.	1.301	1.680	2.015	2.414	2.692	3.286
45.	1.301	1.679	2.014	2.412	2.690	3.281
46.	1.300	1.679	2.013	2.410	2.687	3.277
47.	1.300	1.678	2.012	2.408	2.685	3.273
48.	1.299	1.677	2.011	2.407	2.682	3.269
49.	1.299	1.677	2.010	2.405	2.680	3.265
50.	1.299	1.676	2.009	2.403	2.678	3.261
51.	1.298	1.675	2.008	2.402	2.676	3.258
52.	1.298	1.675	2.007	2.400	2.674	

						3.255
53.	1.298	1.674	2.006	2.399	2.672	3.251
54.	1.297	1.674	2.005	2.397	2.670	3.248
55.	1.297	1.673	2.004	2.396	2.668	3.245
56.	1.297	1.673	2.003	2.395	2.667	3.242
57.	1.297	1.672	2.002	2.394	2.665	3.239
58.	1.296	1.672	2.002	2.392	2.663	3.237
59.	1.296	1.671	2.001	2.391	2.662	3.234
60.	1.296	1.671	2.000	2.390	2.660	3.232
61.	1.296	1.670	2.000	2.389	2.659	3.229
62.	1.295	1.670	1.999	2.388	2.657	3.227
63.	1.295	1.669	1.998	2.387	2.656	3.225
64.	1.295	1.669	1.998	2.386	2.655	3.223
65.	1.295	1.669	1.997	2.385	2.654	3.220
66.	1.295	1.668	1.997	2.384	2.652	3.218
67.	1.294	1.668	1.996	2.383	2.651	3.216
68.	1.294	1.668	1.995	2.382	2.650	3.214
69.	1.294	1.667	1.995	2.382	2.649	3.213
70.	1.294	1.667	1.994	2.381	2.648	3.211
71.	1.294	1.667	1.994	2.380	2.647	3.209
72.	1.293	1.666	1.993	2.379	2.646	3.207
73.	1.293	1.666	1.993	2.379	2.645	3.206
74.	1.293	1.666	1.993	2.378	2.644	3.204
75.	1.293	1.665	1.992	2.377	2.643	3.202
76.	1.293	1.665	1.992	2.376	2.642	3.201
77.	1.293	1.665	1.991	2.376	2.641	3.199
78.	1.292	1.665	1.991	2.375	2.640	3.198
79.	1.292	1.664	1.990	2.374	2.640	3.197
80.	1.292	1.664	1.990	2.374	2.639	3.195
81.	1.292	1.664	1.990	2.373	2.638	3.194
82.	1.292	1.664	1.989	2.373	2.637	3.193
83.	1.292	1.663	1.989	2.372	2.636	3.191
84.	1.292	1.663	1.989	2.372	2.636	

						3.190
85.	1.292	1.663	1.988	2.371	2.635	3.189
86.	1.291	1.663	1.988	2.370	2.634	3.188
87.	1.291	1.663	1.988	2.370	2.634	3.187
88.	1.291	1.662	1.987	2.369	2.633	3.185
89.	1.291	1.662	1.987	2.369	2.632	3.184
90.	1.291	1.662	1.987	2.368	2.632	3.183
91.	1.291	1.662	1.986	2.368	2.631	3.182
92.	1.291	1.662	1.986	2.368	2.630	3.181
93.	1.291	1.661	1.986	2.367	2.630	3.180
94.	1.291	1.661	1.986	2.367	2.629	3.179
95.	1.291	1.661	1.985	2.366	2.629	3.178
96.	1.290	1.661	1.985	2.366	2.628	3.177
97.	1.290	1.661	1.985	2.365	2.627	3.176
98.	1.290	1.661	1.984	2.365	2.627	3.175
99.	1.290	1.660	1.984	2.365	2.626	3.175
100.	1.290	1.660	1.984	2.364	2.626	3.174
∞	1.282	1.645	1.960	2.326	2.576	3.090

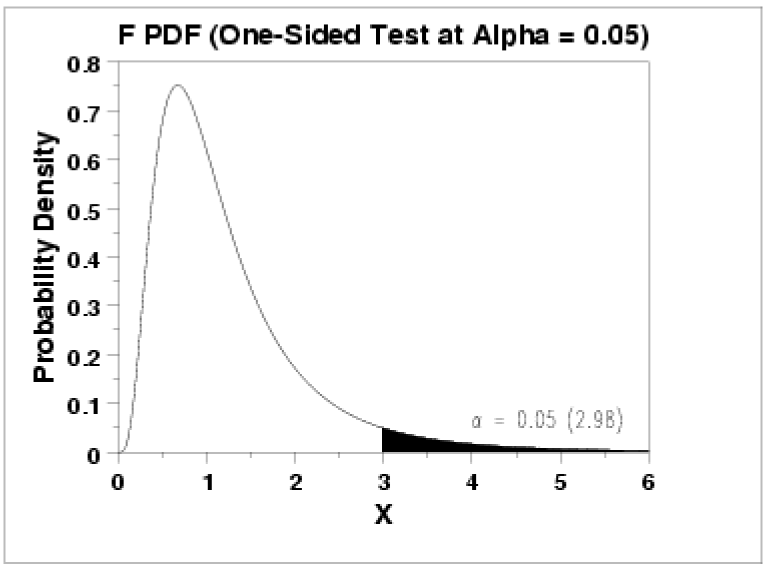
NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.3.6.7.3. Upper Critical Values of the F Distribution

How to Use This Table

This table contains the upper critical values of the F distribution. This table is used for one-sided F tests at the α = 0.05, 0.10, and 0.01 levels.

More specifically, a test statistic is computed with ν_1 and ν_2 degrees of freedom, and the result is compared to this table. For a one-sided test, the null hypothesis is rejected when the test statistic is greater than the tabled value. This is demonstrated with the graph of an F distribution with ν_1 = 10 and ν_2 = 10. The shaded area of the graph indicates the rejection region at the α significance level. Since this is a one-sided test, we have α probability in the upper tail of exceeding the critical value and zero in the lower tail. Because the F distribution is asymmetric, a two-sided test requires a set of tables (not included here) that contain the rejection regions for both the lower and upper tails.



Contents

The following tables for ν_2 from 1 to 100 are included:

1. One sided, 5% significance level, ν_1 = 1 - 10
2. One sided, 5% significance level, ν_1 = 11 - 20
3. One sided, 10% significance level, ν_1 = 1 - 10
4. One sided, 10% significance level, ν_1 = 11 - 20

Upper critical values of the F distribution for ν_1 numerator degrees of freedom and ν_2 denominator degrees of freedom

5% significance level

$$F_{.05}(\nu_1, \nu_2)$$

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10
1	161.448	199.500	215.707	224.583	230.162	233.986	236.768	238.882	240.543	241.882
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385	19.396
3	10.128	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637
8	5.318	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767	2.714	2.671
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348
21	4.325	3.467	3.072	2.840	2.685	2.573	2.488	2.420	2.366	2.321

22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297
23	4.279	3.422	3.028	2.796	2.640	2.528	2.442	2.375	2.320	2.275
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405	2.337	2.282	2.236
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388	2.321	2.265	2.220
27	4.210	3.354	2.960	2.728	2.572	2.459	2.373	2.305	2.250	2.204
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359	2.291	2.236	2.190
29	4.183	3.328	2.934	2.701	2.545	2.432	2.346	2.278	2.223	2.177
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165
31	4.160	3.305	2.911	2.679	2.523	2.409	2.323	2.255	2.199	2.153
32	4.149	3.295	2.901	2.668	2.512	2.399	2.313	2.244	2.189	2.142
33	4.139	3.285	2.892	2.659	2.503	2.389	2.303	2.235	2.179	2.133
34	4.130	3.276	2.883	2.650	2.494	2.380	2.294	2.225	2.170	2.123
35	4.121	3.267	2.874	2.641	2.485	2.372	2.285	2.217	2.161	2.114
36	4.113	3.259	2.866	2.634	2.477	2.364	2.277	2.209	2.153	2.106
37	4.105	3.252	2.859	2.626	2.470	2.356	2.270	2.201	2.145	2.098
38	4.098	3.245	2.852	2.619	2.463	2.349	2.262	2.194	2.138	2.091
39	4.091	3.238	2.845	2.612	2.456	2.342	2.255	2.187	2.131	2.084
40	4.085	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077
41	4.079	3.226	2.833	2.600	2.443	2.330	2.243	2.174	2.118	2.071
42	4.073	3.220	2.827	2.594	2.438	2.324	2.237	2.168	2.112	2.065
43	4.067	3.214	2.822	2.589	2.432	2.318	2.232	2.163	2.106	2.059
44	4.062	3.209	2.816	2.584	2.427	2.313	2.226	2.157	2.101	2.054
45	4.057	3.204	2.812	2.579	2.422	2.308	2.221	2.152	2.096	2.049
46	4.052	3.200	2.807	2.574	2.417	2.304	2.216	2.147	2.091	2.044
47	4.047	3.195	2.802	2.570	2.413	2.299	2.212	2.143	2.086	2.039
48	4.043	3.191	2.798	2.565	2.409	2.295	2.207	2.138	2.082	2.035
49	4.038	3.187	2.794	2.561	2.404	2.290	2.203	2.134	2.077	2.030
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026
51	4.030	3.179	2.786	2.553	2.397	2.283	2.195	2.126	2.069	2.022
52	4.027	3.175	2.783	2.550	2.393	2.279	2.192	2.122	2.066	2.018
53	4.023	3.172	2.779	2.546	2.389	2.275	2.188	2.119	2.062	2.015

54	4.020	3.168	2.776	2.543	2.386	2.272	2.185	2.115	2.059	2.011
55	4.016	3.165	2.773	2.540	2.383	2.269	2.181	2.112	2.055	2.008
56	4.013	3.162	2.769	2.537	2.380	2.266	2.178	2.109	2.052	2.005
57	4.010	3.159	2.766	2.534	2.377	2.263	2.175	2.106	2.049	2.001
58	4.007	3.156	2.764	2.531	2.374	2.260	2.172	2.103	2.046	1.998
59	4.004	3.153	2.761	2.528	2.371	2.257	2.169	2.100	2.043	1.995
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993
61	3.998	3.148	2.755	2.523	2.366	2.251	2.164	2.094	2.037	1.990
62	3.996	3.145	2.753	2.520	2.363	2.249	2.161	2.092	2.035	1.987
63	3.993	3.143	2.751	2.518	2.361	2.246	2.159	2.089	2.032	1.985
64	3.991	3.140	2.748	2.515	2.358	2.244	2.156	2.087	2.030	1.982
65	3.989	3.138	2.746	2.513	2.356	2.242	2.154	2.084	2.027	1.980
66	3.986	3.136	2.744	2.511	2.354	2.239	2.152	2.082	2.025	1.977
67	3.984	3.134	2.742	2.509	2.352	2.237	2.150	2.080	2.023	1.975
68	3.982	3.132	2.740	2.507	2.350	2.235	2.148	2.078	2.021	1.973
69	3.980	3.130	2.737	2.505	2.348	2.233	2.145	2.076	2.019	1.971
70	3.978	3.128	2.736	2.503	2.346	2.231	2.143	2.074	2.017	1.969
71	3.976	3.126	2.734	2.501	2.344	2.229	2.142	2.072	2.015	1.967
72	3.974	3.124	2.732	2.499	2.342	2.227	2.140	2.070	2.013	1.965
73	3.972	3.122	2.730	2.497	2.340	2.226	2.138	2.068	2.011	1.963
74	3.970	3.120	2.728	2.495	2.338	2.224	2.136	2.066	2.009	1.961
75	3.968	3.119	2.727	2.494	2.337	2.222	2.134	2.064	2.007	1.959
76	3.967	3.117	2.725	2.492	2.335	2.220	2.133	2.063	2.006	1.958
77	3.965	3.115	2.723	2.490	2.333	2.219	2.131	2.061	2.004	1.956
78	3.963	3.114	2.722	2.489	2.332	2.217	2.129	2.059	2.002	1.954
79	3.962	3.112	2.720	2.487	2.330	2.216	2.128	2.058	2.001	1.953
80	3.960	3.111	2.719	2.486	2.329	2.214	2.126	2.056	1.999	1.951
81	3.959	3.109	2.717	2.484	2.327	2.213	2.125	2.055	1.998	1.950
82	3.957	3.108	2.716	2.483	2.326	2.211	2.123	2.053	1.996	1.948
83	3.956	3.107	2.715	2.482	2.324	2.210	2.122	2.052	1.995	1.947
84	3.955	3.105	2.713	2.480	2.323	2.209	2.121	2.051	1.993	1.945
85	3.953	3.104	2.712	2.479	2.322	2.207	2.119	2.049	1.992	1.944

86	3.952	3.103	2.711	2.478	2.321	2.206	2.118	2.048	1.991	1.943
87	3.951	3.101	2.709	2.476	2.319	2.205	2.117	2.047	1.989	1.941
88	3.949	3.100	2.708	2.475	2.318	2.203	2.115	2.045	1.988	1.940
89	3.948	3.099	2.707	2.474	2.317	2.202	2.114	2.044	1.987	1.939
90	3.947	3.098	2.706	2.473	2.316	2.201	2.113	2.043	1.986	1.938
91	3.946	3.097	2.705	2.472	2.315	2.200	2.112	2.042	1.984	1.936
92	3.945	3.095	2.704	2.471	2.313	2.199	2.111	2.041	1.983	1.935
93	3.943	3.094	2.703	2.470	2.312	2.198	2.110	2.040	1.982	1.934
94	3.942	3.093	2.701	2.469	2.311	2.197	2.109	2.038	1.981	1.933
95	3.941	3.092	2.700	2.467	2.310	2.196	2.108	2.037	1.980	1.932
96	3.940	3.091	2.699	2.466	2.309	2.195	2.106	2.036	1.979	1.931
97	3.939	3.090	2.698	2.465	2.308	2.194	2.105	2.035	1.978	1.930
98	3.938	3.089	2.697	2.465	2.307	2.193	2.104	2.034	1.977	1.929
99	3.937	3.088	2.696	2.464	2.306	2.192	2.103	2.033	1.976	1.928
100	3.936	3.087	2.696	2.463	2.305	2.191	2.103	2.032	1.975	1.927

ν_2 \ ν_1	11	12	13	14	15	16	17	18	19	20
1	242.983	243.906	244.690	245.364	245.950	246.464	246.918	247.323	247.686	248.013
2	19.405	19.413	19.419	19.424	19.429	19.433	19.437	19.440	19.443	19.446
3	8.763	8.745	8.729	8.715	8.703	8.692	8.683	8.675	8.667	8.660
4	5.936	5.912	5.891	5.873	5.858	5.844	5.832	5.821	5.811	5.803
5	4.704	4.678	4.655	4.636	4.619	4.604	4.590	4.579	4.568	4.558
6	4.027	4.000	3.976	3.956	3.938	3.922	3.908	3.896	3.884	3.874
7	3.603	3.575	3.550	3.529	3.511	3.494	3.480	3.467	3.455	3.445
8	3.313	3.284	3.259	3.237	3.218	3.202	3.187	3.173	3.161	3.150
9	3.102	3.073	3.048	3.025	3.006	2.989	2.974	2.960	2.948	2.936
10	2.943	2.913	2.887	2.865	2.845	2.828	2.812	2.798	2.785	2.774
11	2.818	2.788	2.761	2.739	2.719	2.701	2.685	2.671	2.658	2.646
12	2.717	2.687	2.660	2.637	2.617	2.599	2.583	2.568	2.555	2.544
13	2.635	2.604	2.577	2.554	2.533	2.515	2.499	2.484	2.471	2.459

14	2.565	2.534	2.507	2.484	2.463	2.445	2.428	2.413	2.400	2.388
15	2.507	2.475	2.448	2.424	2.403	2.385	2.368	2.353	2.340	2.328
16	2.456	2.425	2.397	2.373	2.352	2.333	2.317	2.302	2.288	2.276
17	2.413	2.381	2.353	2.329	2.308	2.289	2.272	2.257	2.243	2.230
18	2.374	2.342	2.314	2.290	2.269	2.250	2.233	2.217	2.203	2.191
19	2.340	2.308	2.280	2.256	2.234	2.215	2.198	2.182	2.168	2.155
20	2.310	2.278	2.250	2.225	2.203	2.184	2.167	2.151	2.137	2.124
21	2.283	2.250	2.222	2.197	2.176	2.156	2.139	2.123	2.109	2.096
22	2.259	2.226	2.198	2.173	2.151	2.131	2.114	2.098	2.084	2.071
23	2.236	2.204	2.175	2.150	2.128	2.109	2.091	2.075	2.061	2.048
24	2.216	2.183	2.155	2.130	2.108	2.088	2.070	2.054	2.040	2.027
25	2.198	2.165	2.136	2.111	2.089	2.069	2.051	2.035	2.021	2.007
26	2.181	2.148	2.119	2.094	2.072	2.052	2.034	2.018	2.003	1.990
27	2.166	2.132	2.103	2.078	2.056	2.036	2.018	2.002	1.987	1.974
28	2.151	2.118	2.089	2.064	2.041	2.021	2.003	1.987	1.972	1.959
29	2.138	2.104	2.075	2.050	2.027	2.007	1.989	1.973	1.958	1.945
30	2.126	2.092	2.063	2.037	2.015	1.995	1.976	1.960	1.945	1.932
31	2.114	2.080	2.051	2.026	2.003	1.983	1.965	1.948	1.933	1.920
32	2.103	2.070	2.040	2.015	1.992	1.972	1.953	1.937	1.922	1.908
33	2.093	2.060	2.030	2.004	1.982	1.961	1.943	1.926	1.911	1.898
34	2.084	2.050	2.021	1.995	1.972	1.952	1.933	1.917	1.902	1.888
35	2.075	2.041	2.012	1.986	1.963	1.942	1.924	1.907	1.892	1.878
36	2.067	2.033	2.003	1.977	1.954	1.934	1.915	1.899	1.883	1.870
37	2.059	2.025	1.995	1.969	1.946	1.926	1.907	1.890	1.875	1.861
38	2.051	2.017	1.988	1.962	1.939	1.918	1.899	1.883	1.867	1.853
39	2.044	2.010	1.981	1.954	1.931	1.911	1.892	1.875	1.860	1.846
40	2.038	2.003	1.974	1.948	1.924	1.904	1.885	1.868	1.853	1.839
41	2.031	1.997	1.967	1.941	1.918	1.897	1.879	1.862	1.846	1.832
42	2.025	1.991	1.961	1.935	1.912	1.891	1.872	1.855	1.840	1.826
43	2.020	1.985	1.955	1.929	1.906	1.885	1.866	1.849	1.834	1.820
44	2.014	1.980	1.950	1.924	1.900	1.879	1.861	1.844	1.828	1.814
45	2.009	1.974	1.945	1.918	1.895	1.874	1.855	1.838	1.823	1.808

46	2.004	1.969	1.940	1.913	1.890	1.869	1.850	1.833	1.817	1.803
47	1.999	1.965	1.935	1.908	1.885	1.864	1.845	1.828	1.812	1.798
48	1.995	1.960	1.930	1.904	1.880	1.859	1.840	1.823	1.807	1.793
49	1.990	1.956	1.926	1.899	1.876	1.855	1.836	1.819	1.803	1.789
50	1.986	1.952	1.921	1.895	1.871	1.850	1.831	1.814	1.798	1.784
51	1.982	1.947	1.917	1.891	1.867	1.846	1.827	1.810	1.794	1.780
52	1.978	1.944	1.913	1.887	1.863	1.842	1.823	1.806	1.790	1.776
53	1.975	1.940	1.910	1.883	1.859	1.838	1.819	1.802	1.786	1.772
54	1.971	1.936	1.906	1.879	1.856	1.835	1.816	1.798	1.782	1.768
55	1.968	1.933	1.903	1.876	1.852	1.831	1.812	1.795	1.779	1.764
56	1.964	1.930	1.899	1.873	1.849	1.828	1.809	1.791	1.775	1.761
57	1.961	1.926	1.896	1.869	1.846	1.824	1.805	1.788	1.772	1.757
58	1.958	1.923	1.893	1.866	1.842	1.821	1.802	1.785	1.769	1.754
59	1.955	1.920	1.890	1.863	1.839	1.818	1.799	1.781	1.766	1.751
60	1.952	1.917	1.887	1.860	1.836	1.815	1.796	1.778	1.763	1.748
61	1.949	1.915	1.884	1.857	1.834	1.812	1.793	1.776	1.760	1.745
62	1.947	1.912	1.882	1.855	1.831	1.809	1.790	1.773	1.757	1.742
63	1.944	1.909	1.879	1.852	1.828	1.807	1.787	1.770	1.754	1.739
64	1.942	1.907	1.876	1.849	1.826	1.804	1.785	1.767	1.751	1.737
65	1.939	1.904	1.874	1.847	1.823	1.802	1.782	1.765	1.749	1.734
66	1.937	1.902	1.871	1.845	1.821	1.799	1.780	1.762	1.746	1.732
67	1.935	1.900	1.869	1.842	1.818	1.797	1.777	1.760	1.744	1.729
68	1.932	1.897	1.867	1.840	1.816	1.795	1.775	1.758	1.742	1.727
69	1.930	1.895	1.865	1.838	1.814	1.792	1.773	1.755	1.739	1.725
70	1.928	1.893	1.863	1.836	1.812	1.790	1.771	1.753	1.737	1.722
71	1.926	1.891	1.861	1.834	1.810	1.788	1.769	1.751	1.735	1.720
72	1.924	1.889	1.859	1.832	1.808	1.786	1.767	1.749	1.733	1.718
73	1.922	1.887	1.857	1.830	1.806	1.784	1.765	1.747	1.731	1.716
74	1.921	1.885	1.855	1.828	1.804	1.782	1.763	1.745	1.729	1.714
75	1.919	1.884	1.853	1.826	1.802	1.780	1.761	1.743	1.727	1.712
76	1.917	1.882	1.851	1.824	1.800	1.778	1.759	1.741	1.725	1.710
77	1.915	1.880	1.849	1.822	1.798	1.777	1.757	1.739	1.723	1.708

78	1.914	1.878	1.848	1.821	1.797	1.775	1.755	1.738	1.721	1.707
79	1.912	1.877	1.846	1.819	1.795	1.773	1.754	1.736	1.720	1.705
80	1.910	1.875	1.845	1.817	1.793	1.772	1.752	1.734	1.718	1.703
81	1.909	1.874	1.843	1.816	1.792	1.770	1.750	1.733	1.716	1.702
82	1.907	1.872	1.841	1.814	1.790	1.768	1.749	1.731	1.715	1.700
83	1.906	1.871	1.840	1.813	1.789	1.767	1.747	1.729	1.713	1.698
84	1.905	1.869	1.838	1.811	1.787	1.765	1.746	1.728	1.712	1.697
85	1.903	1.868	1.837	1.810	1.786	1.764	1.744	1.726	1.710	1.695
86	1.902	1.867	1.836	1.808	1.784	1.762	1.743	1.725	1.709	1.694
87	1.900	1.865	1.834	1.807	1.783	1.761	1.741	1.724	1.707	1.692
88	1.899	1.864	1.833	1.806	1.782	1.760	1.740	1.722	1.706	1.691
89	1.898	1.863	1.832	1.804	1.780	1.758	1.739	1.721	1.705	1.690
90	1.897	1.861	1.830	1.803	1.779	1.757	1.737	1.720	1.703	1.688
91	1.895	1.860	1.829	1.802	1.778	1.756	1.736	1.718	1.702	1.687
92	1.894	1.859	1.828	1.801	1.776	1.755	1.735	1.717	1.701	1.686
93	1.893	1.858	1.827	1.800	1.775	1.753	1.734	1.716	1.699	1.684
94	1.892	1.857	1.826	1.798	1.774	1.752	1.733	1.715	1.698	1.683
95	1.891	1.856	1.825	1.797	1.773	1.751	1.731	1.713	1.697	1.682
96	1.890	1.854	1.823	1.796	1.772	1.750	1.730	1.712	1.696	1.681
97	1.889	1.853	1.822	1.795	1.771	1.749	1.729	1.711	1.695	1.680
98	1.888	1.852	1.821	1.794	1.770	1.748	1.728	1.710	1.694	1.679
99	1.887	1.851	1.820	1.793	1.769	1.747	1.727	1.709	1.693	1.678
100	1.886	1.850	1.819	1.792	1.768	1.746	1.726	1.708	1.691	1.676

Upper critical values of the F distribution for ν_1 numerator degrees of freedom and ν_2 denominator degrees of freedom

10% significance level

$$F_{.10}(\nu_1, \nu_2)$$

\ ν_1	1	2	3	4	5	6	7	8	9	10

ν_2										
1	39.863	49.500	53.593	55.833	57.240	58.204	58.906	59.439	59.858	60.195
2	8.526	9.000	9.162	9.243	9.293	9.326	9.349	9.367	9.381	9.392
3	5.538	5.462	5.391	5.343	5.309	5.285	5.266	5.252	5.240	5.230
4	4.545	4.325	4.191	4.107	4.051	4.010	3.979	3.955	3.936	3.920
5	4.060	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.316	3.297
6	3.776	3.463	3.289	3.181	3.108	3.055	3.014	2.983	2.958	2.937
7	3.589	3.257	3.074	2.961	2.883	2.827	2.785	2.752	2.725	2.703
8	3.458	3.113	2.924	2.806	2.726	2.668	2.624	2.589	2.561	2.538
9	3.360	3.006	2.813	2.693	2.611	2.551	2.505	2.469	2.440	2.416
10	3.285	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.347	2.323
11	3.225	2.860	2.660	2.536	2.451	2.389	2.342	2.304	2.274	2.248
12	3.177	2.807	2.606	2.480	2.394	2.331	2.283	2.245	2.214	2.188
13	3.136	2.763	2.560	2.434	2.347	2.283	2.234	2.195	2.164	2.138
14	3.102	2.726	2.522	2.395	2.307	2.243	2.193	2.154	2.122	2.095
15	3.073	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.086	2.059
16	3.048	2.668	2.462	2.333	2.244	2.178	2.128	2.088	2.055	2.028
17	3.026	2.645	2.437	2.308	2.218	2.152	2.102	2.061	2.028	2.001
18	3.007	2.624	2.416	2.286	2.196	2.130	2.079	2.038	2.005	1.977
19	2.990	2.606	2.397	2.266	2.176	2.109	2.058	2.017	1.984	1.956
20	2.975	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.965	1.937
21	2.961	2.575	2.365	2.233	2.142	2.075	2.023	1.982	1.948	1.920
22	2.949	2.561	2.351	2.219	2.128	2.060	2.008	1.967	1.933	1.904
23	2.937	2.549	2.339	2.207	2.115	2.047	1.995	1.953	1.919	1.890
24	2.927	2.538	2.327	2.195	2.103	2.035	1.983	1.941	1.906	1.877
25	2.918	2.528	2.317	2.184	2.092	2.024	1.971	1.929	1.895	1.866
26	2.909	2.519	2.307	2.174	2.082	2.014	1.961	1.919	1.884	1.855
27	2.901	2.511	2.299	2.165	2.073	2.005	1.952	1.909	1.874	1.845
28	2.894	2.503	2.291	2.157	2.064	1.996	1.943	1.900	1.865	1.836
29	2.887	2.495	2.283	2.149	2.057	1.988	1.935	1.892	1.857	1.827
30	2.881	2.489	2.276	2.142	2.049	1.980	1.927	1.884	1.849	1.819
31	2.875	2.482	2.270	2.136	2.042	1.973	1.920	1.877	1.842	1.812

32	2.869	2.477	2.263	2.129	2.036	1.967	1.913	1.870	1.835	1.805
33	2.864	2.471	2.258	2.123	2.030	1.961	1.907	1.864	1.828	1.799
34	2.859	2.466	2.252	2.118	2.024	1.955	1.901	1.858	1.822	1.793
35	2.855	2.461	2.247	2.113	2.019	1.950	1.896	1.852	1.817	1.787
36	2.850	2.456	2.243	2.108	2.014	1.945	1.891	1.847	1.811	1.781
37	2.846	2.452	2.238	2.103	2.009	1.940	1.886	1.842	1.806	1.776
38	2.842	2.448	2.234	2.099	2.005	1.935	1.881	1.838	1.802	1.772
39	2.839	2.444	2.230	2.095	2.001	1.931	1.877	1.833	1.797	1.767
40	2.835	2.440	2.226	2.091	1.997	1.927	1.873	1.829	1.793	1.763
41	2.832	2.437	2.222	2.087	1.993	1.923	1.869	1.825	1.789	1.759
42	2.829	2.434	2.219	2.084	1.989	1.919	1.865	1.821	1.785	1.755
43	2.826	2.430	2.216	2.080	1.986	1.916	1.861	1.817	1.781	1.751
44	2.823	2.427	2.213	2.077	1.983	1.913	1.858	1.814	1.778	1.747
45	2.820	2.425	2.210	2.074	1.980	1.909	1.855	1.811	1.774	1.744
46	2.818	2.422	2.207	2.071	1.977	1.906	1.852	1.808	1.771	1.741
47	2.815	2.419	2.204	2.068	1.974	1.903	1.849	1.805	1.768	1.738
48	2.813	2.417	2.202	2.066	1.971	1.901	1.846	1.802	1.765	1.735
49	2.811	2.414	2.199	2.063	1.968	1.898	1.843	1.799	1.763	1.732
50	2.809	2.412	2.197	2.061	1.966	1.895	1.840	1.796	1.760	1.729
51	2.807	2.410	2.194	2.058	1.964	1.893	1.838	1.794	1.757	1.727
52	2.805	2.408	2.192	2.056	1.961	1.891	1.836	1.791	1.755	1.724
53	2.803	2.406	2.190	2.054	1.959	1.888	1.833	1.789	1.752	1.722
54	2.801	2.404	2.188	2.052	1.957	1.886	1.831	1.787	1.750	1.719
55	2.799	2.402	2.186	2.050	1.955	1.884	1.829	1.785	1.748	1.717
56	2.797	2.400	2.184	2.048	1.953	1.882	1.827	1.782	1.746	1.715
57	2.796	2.398	2.182	2.046	1.951	1.880	1.825	1.780	1.744	1.713
58	2.794	2.396	2.181	2.044	1.949	1.878	1.823	1.779	1.742	1.711
59	2.793	2.395	2.179	2.043	1.947	1.876	1.821	1.777	1.740	1.709
60	2.791	2.393	2.177	2.041	1.946	1.875	1.819	1.775	1.738	1.707
61	2.790	2.392	2.176	2.039	1.944	1.873	1.818	1.773	1.736	1.705
62	2.788	2.390	2.174	2.038	1.942	1.871	1.816	1.771	1.735	1.703
63	2.787	2.389	2.173	2.036	1.941	1.870	1.814	1.770	1.733	1.702

64	2.786	2.387	2.171	2.035	1.939	1.868	1.813	1.768	1.731	1.700
65	2.784	2.386	2.170	2.033	1.938	1.867	1.811	1.767	1.730	1.699
66	2.783	2.385	2.169	2.032	1.937	1.865	1.810	1.765	1.728	1.697
67	2.782	2.384	2.167	2.031	1.935	1.864	1.808	1.764	1.727	1.696
68	2.781	2.382	2.166	2.029	1.934	1.863	1.807	1.762	1.725	1.694
69	2.780	2.381	2.165	2.028	1.933	1.861	1.806	1.761	1.724	1.693
70	2.779	2.380	2.164	2.027	1.931	1.860	1.804	1.760	1.723	1.691
71	2.778	2.379	2.163	2.026	1.930	1.859	1.803	1.758	1.721	1.690
72	2.777	2.378	2.161	2.025	1.929	1.858	1.802	1.757	1.720	1.689
73	2.776	2.377	2.160	2.024	1.928	1.856	1.801	1.756	1.719	1.687
74	2.775	2.376	2.159	2.022	1.927	1.855	1.800	1.755	1.718	1.686
75	2.774	2.375	2.158	2.021	1.926	1.854	1.798	1.754	1.716	1.685
76	2.773	2.374	2.157	2.020	1.925	1.853	1.797	1.752	1.715	1.684
77	2.772	2.373	2.156	2.019	1.924	1.852	1.796	1.751	1.714	1.683
78	2.771	2.372	2.155	2.018	1.923	1.851	1.795	1.750	1.713	1.682
79	2.770	2.371	2.154	2.017	1.922	1.850	1.794	1.749	1.712	1.681
80	2.769	2.370	2.154	2.016	1.921	1.849	1.793	1.748	1.711	1.680
81	2.769	2.369	2.153	2.016	1.920	1.848	1.792	1.747	1.710	1.679
82	2.768	2.368	2.152	2.015	1.919	1.847	1.791	1.746	1.709	1.678
83	2.767	2.368	2.151	2.014	1.918	1.846	1.790	1.745	1.708	1.677
84	2.766	2.367	2.150	2.013	1.917	1.845	1.790	1.744	1.707	1.676
85	2.765	2.366	2.149	2.012	1.916	1.845	1.789	1.744	1.706	1.675
86	2.765	2.365	2.149	2.011	1.915	1.844	1.788	1.743	1.705	1.674
87	2.764	2.365	2.148	2.011	1.915	1.843	1.787	1.742	1.705	1.673
88	2.763	2.364	2.147	2.010	1.914	1.842	1.786	1.741	1.704	1.672
89	2.763	2.363	2.146	2.009	1.913	1.841	1.785	1.740	1.703	1.671
90	2.762	2.363	2.146	2.008	1.912	1.841	1.785	1.739	1.702	1.670
91	2.761	2.362	2.145	2.008	1.912	1.840	1.784	1.739	1.701	1.670
92	2.761	2.361	2.144	2.007	1.911	1.839	1.783	1.738	1.701	1.669
93	2.760	2.361	2.144	2.006	1.910	1.838	1.782	1.737	1.700	1.668
94	2.760	2.360	2.143	2.006	1.910	1.838	1.782	1.736	1.699	1.667
95	2.759	2.359	2.142	2.005	1.909	1.837	1.781	1.736	1.698	1.667

96	2.759	2.359	2.142	2.004	1.908	1.836	1.780	1.735	1.698	1.666
97	2.758	2.358	2.141	2.004	1.908	1.836	1.780	1.734	1.697	1.665
98	2.757	2.358	2.141	2.003	1.907	1.835	1.779	1.734	1.696	1.665
99	2.757	2.357	2.140	2.003	1.906	1.835	1.778	1.733	1.696	1.664
100	2.756	2.356	2.139	2.002	1.906	1.834	1.778	1.732	1.695	1.663

ν_2 \ ν_1	11	12	13	14	15	16	17	18	19	20
1	60.473	60.705	60.903	61.073	61.220	61.350	61.464	61.566	61.658	61.740
2	9.401	9.408	9.415	9.420	9.425	9.429	9.433	9.436	9.439	9.441
3	5.222	5.216	5.210	5.205	5.200	5.196	5.193	5.190	5.187	5.184
4	3.907	3.896	3.886	3.878	3.870	3.864	3.858	3.853	3.849	3.844
5	3.282	3.268	3.257	3.247	3.238	3.230	3.223	3.217	3.212	3.207
6	2.920	2.905	2.892	2.881	2.871	2.863	2.855	2.848	2.842	2.836
7	2.684	2.668	2.654	2.643	2.632	2.623	2.615	2.607	2.601	2.595
8	2.519	2.502	2.488	2.475	2.464	2.455	2.446	2.438	2.431	2.425
9	2.396	2.379	2.364	2.351	2.340	2.329	2.320	2.312	2.305	2.298
10	2.302	2.284	2.269	2.255	2.244	2.233	2.224	2.215	2.208	2.201
11	2.227	2.209	2.193	2.179	2.167	2.156	2.147	2.138	2.130	2.123
12	2.166	2.147	2.131	2.117	2.105	2.094	2.084	2.075	2.067	2.060
13	2.116	2.097	2.080	2.066	2.053	2.042	2.032	2.023	2.014	2.007
14	2.073	2.054	2.037	2.022	2.010	1.998	1.988	1.978	1.970	1.962
15	2.037	2.017	2.000	1.985	1.972	1.961	1.950	1.941	1.932	1.924
16	2.005	1.985	1.968	1.953	1.940	1.928	1.917	1.908	1.899	1.891
17	1.978	1.958	1.940	1.925	1.912	1.900	1.889	1.879	1.870	1.862
18	1.954	1.933	1.916	1.900	1.887	1.875	1.864	1.854	1.845	1.837
19	1.932	1.912	1.894	1.878	1.865	1.852	1.841	1.831	1.822	1.814
20	1.913	1.892	1.875	1.859	1.845	1.833	1.821	1.811	1.802	1.794
21	1.896	1.875	1.857	1.841	1.827	1.815	1.803	1.793	1.784	1.776
22	1.880	1.859	1.841	1.825	1.811	1.798	1.787	1.777	1.768	1.759
23	1.866	1.845	1.827	1.811	1.796	1.784	1.772	1.762	1.753	1.744

24	1.853	1.832	1.814	1.797	1.783	1.770	1.759	1.748	1.739	1.730
25	1.841	1.820	1.802	1.785	1.771	1.758	1.746	1.736	1.726	1.718
26	1.830	1.809	1.790	1.774	1.760	1.747	1.735	1.724	1.715	1.706
27	1.820	1.799	1.780	1.764	1.749	1.736	1.724	1.714	1.704	1.695
28	1.811	1.790	1.771	1.754	1.740	1.726	1.715	1.704	1.694	1.685
29	1.802	1.781	1.762	1.745	1.731	1.717	1.705	1.695	1.685	1.676
30	1.794	1.773	1.754	1.737	1.722	1.709	1.697	1.686	1.676	1.667
31	1.787	1.765	1.746	1.729	1.714	1.701	1.689	1.678	1.668	1.659
32	1.780	1.758	1.739	1.722	1.707	1.694	1.682	1.671	1.661	1.652
33	1.773	1.751	1.732	1.715	1.700	1.687	1.675	1.664	1.654	1.645
34	1.767	1.745	1.726	1.709	1.694	1.680	1.668	1.657	1.647	1.638
35	1.761	1.739	1.720	1.703	1.688	1.674	1.662	1.651	1.641	1.632
36	1.756	1.734	1.715	1.697	1.682	1.669	1.656	1.645	1.635	1.626
37	1.751	1.729	1.709	1.692	1.677	1.663	1.651	1.640	1.630	1.620
38	1.746	1.724	1.704	1.687	1.672	1.658	1.646	1.635	1.624	1.615
39	1.741	1.719	1.700	1.682	1.667	1.653	1.641	1.630	1.619	1.610
40	1.737	1.715	1.695	1.678	1.662	1.649	1.636	1.625	1.615	1.605
41	1.733	1.710	1.691	1.673	1.658	1.644	1.632	1.620	1.610	1.601
42	1.729	1.706	1.687	1.669	1.654	1.640	1.628	1.616	1.606	1.596
43	1.725	1.703	1.683	1.665	1.650	1.636	1.624	1.612	1.602	1.592
44	1.721	1.699	1.679	1.662	1.646	1.632	1.620	1.608	1.598	1.588
45	1.718	1.695	1.676	1.658	1.643	1.629	1.616	1.605	1.594	1.585
46	1.715	1.692	1.672	1.655	1.639	1.625	1.613	1.601	1.591	1.581
47	1.712	1.689	1.669	1.652	1.636	1.622	1.609	1.598	1.587	1.578
48	1.709	1.686	1.666	1.648	1.633	1.619	1.606	1.594	1.584	1.574
49	1.706	1.683	1.663	1.645	1.630	1.616	1.603	1.591	1.581	1.571
50	1.703	1.680	1.660	1.643	1.627	1.613	1.600	1.588	1.578	1.568
51	1.700	1.677	1.658	1.640	1.624	1.610	1.597	1.586	1.575	1.565
52	1.698	1.675	1.655	1.637	1.621	1.607	1.594	1.583	1.572	1.562
53	1.695	1.672	1.652	1.635	1.619	1.605	1.592	1.580	1.570	1.560
54	1.693	1.670	1.650	1.632	1.616	1.602	1.589	1.578	1.567	1.557
55	1.691	1.668	1.648	1.630	1.614	1.600	1.587	1.575	1.564	1.555

56	1.688	1.666	1.645	1.628	1.612	1.597	1.585	1.573	1.562	1.552
57	1.686	1.663	1.643	1.625	1.610	1.595	1.582	1.571	1.560	1.550
58	1.684	1.661	1.641	1.623	1.607	1.593	1.580	1.568	1.558	1.548
59	1.682	1.659	1.639	1.621	1.605	1.591	1.578	1.566	1.555	1.546
60	1.680	1.657	1.637	1.619	1.603	1.589	1.576	1.564	1.553	1.543
61	1.679	1.656	1.635	1.617	1.601	1.587	1.574	1.562	1.551	1.541
62	1.677	1.654	1.634	1.616	1.600	1.585	1.572	1.560	1.549	1.540
63	1.675	1.652	1.632	1.614	1.598	1.583	1.570	1.558	1.548	1.538
64	1.673	1.650	1.630	1.612	1.596	1.582	1.569	1.557	1.546	1.536
65	1.672	1.649	1.628	1.610	1.594	1.580	1.567	1.555	1.544	1.534
66	1.670	1.647	1.627	1.609	1.593	1.578	1.565	1.553	1.542	1.532
67	1.669	1.646	1.625	1.607	1.591	1.577	1.564	1.552	1.541	1.531
68	1.667	1.644	1.624	1.606	1.590	1.575	1.562	1.550	1.539	1.529
69	1.666	1.643	1.622	1.604	1.588	1.574	1.560	1.548	1.538	1.527
70	1.665	1.641	1.621	1.603	1.587	1.572	1.559	1.547	1.536	1.526
71	1.663	1.640	1.619	1.601	1.585	1.571	1.557	1.545	1.535	1.524
72	1.662	1.639	1.618	1.600	1.584	1.569	1.556	1.544	1.533	1.523
73	1.661	1.637	1.617	1.599	1.583	1.568	1.555	1.543	1.532	1.522
74	1.659	1.636	1.616	1.597	1.581	1.567	1.553	1.541	1.530	1.520
75	1.658	1.635	1.614	1.596	1.580	1.565	1.552	1.540	1.529	1.519
76	1.657	1.634	1.613	1.595	1.579	1.564	1.551	1.539	1.528	1.518
77	1.656	1.632	1.612	1.594	1.578	1.563	1.550	1.538	1.527	1.516
78	1.655	1.631	1.611	1.593	1.576	1.562	1.548	1.536	1.525	1.515
79	1.654	1.630	1.610	1.592	1.575	1.561	1.547	1.535	1.524	1.514
80	1.653	1.629	1.609	1.590	1.574	1.559	1.546	1.534	1.523	1.513
81	1.652	1.628	1.608	1.589	1.573	1.558	1.545	1.533	1.522	1.512
82	1.651	1.627	1.607	1.588	1.572	1.557	1.544	1.532	1.521	1.511
83	1.650	1.626	1.606	1.587	1.571	1.556	1.543	1.531	1.520	1.509
84	1.649	1.625	1.605	1.586	1.570	1.555	1.542	1.530	1.519	1.508
85	1.648	1.624	1.604	1.585	1.569	1.554	1.541	1.529	1.518	1.507
86	1.647	1.623	1.603	1.584	1.568	1.553	1.540	1.528	1.517	1.506
87	1.646	1.622	1.602	1.583	1.567	1.552	1.539	1.527	1.516	1.505

88	1.645	1.622	1.601	1.583	1.566	1.551	1.538	1.526	1.515	1.504
89	1.644	1.621	1.600	1.582	1.565	1.550	1.537	1.525	1.514	1.503
90	1.643	1.620	1.599	1.581	1.564	1.550	1.536	1.524	1.513	1.503
91	1.643	1.619	1.598	1.580	1.564	1.549	1.535	1.523	1.512	1.502
92	1.642	1.618	1.598	1.579	1.563	1.548	1.534	1.522	1.511	1.501
93	1.641	1.617	1.597	1.578	1.562	1.547	1.534	1.521	1.510	1.500
94	1.640	1.617	1.596	1.578	1.561	1.546	1.533	1.521	1.509	1.499
95	1.640	1.616	1.595	1.577	1.560	1.545	1.532	1.520	1.509	1.498
96	1.639	1.615	1.594	1.576	1.560	1.545	1.531	1.519	1.508	1.497
97	1.638	1.614	1.594	1.575	1.559	1.544	1.530	1.518	1.507	1.497
98	1.637	1.614	1.593	1.575	1.558	1.543	1.530	1.517	1.506	1.496
99	1.637	1.613	1.592	1.574	1.557	1.542	1.529	1.517	1.505	1.495
100	1.636	1.612	1.592	1.573	1.557	1.542	1.528	1.516	1.505	1.494

Upper critical values of the F distribution for ν_1 numerator degrees of freedom and ν_2 denominator degrees of freedom

1% significance level

$$F_{.01}(\nu_1, \nu_2)$$

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10
1	4052.19	4999.52	5403.34	5624.62	5763.65	5858.97	5928.33	5981.10	6022.50	6055.85
2	98.502	99.000	99.166	99.249	99.300	99.333	99.356	99.374	99.388	99.399
3	34.116	30.816	29.457	28.710	28.237	27.911	27.672	27.489	27.345	27.229
4	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659	14.546
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158	10.051
6	13.745	10.925	9.780	9.148	8.746	8.466	8.260	8.102	7.976	7.874
7	12.246	9.547	8.451	7.847	7.460	7.191	6.993	6.840	6.719	6.620
8	11.259	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.911	5.814
9	10.561	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.351	5.257

10	10.044	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849
11	9.646	7.206	6.217	5.668	5.316	5.069	4.886	4.744	4.632	4.539
12	9.330	6.927	5.953	5.412	5.064	4.821	4.640	4.499	4.388	4.296
13	9.074	6.701	5.739	5.205	4.862	4.620	4.441	4.302	4.191	4.100
14	8.862	6.515	5.564	5.035	4.695	4.456	4.278	4.140	4.030	3.939
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805
16	8.531	6.226	5.292	4.773	4.437	4.202	4.026	3.890	3.780	3.691
17	8.400	6.112	5.185	4.669	4.336	4.102	3.927	3.791	3.682	3.593
18	8.285	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.597	3.508
19	8.185	5.926	5.010	4.500	4.171	3.939	3.765	3.631	3.523	3.434
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368
21	8.017	5.780	4.874	4.369	4.042	3.812	3.640	3.506	3.398	3.310
22	7.945	5.719	4.817	4.313	3.988	3.758	3.587	3.453	3.346	3.258
23	7.881	5.664	4.765	4.264	3.939	3.710	3.539	3.406	3.299	3.211
24	7.823	5.614	4.718	4.218	3.895	3.667	3.496	3.363	3.256	3.168
25	7.770	5.568	4.675	4.177	3.855	3.627	3.457	3.324	3.217	3.129
26	7.721	5.526	4.637	4.140	3.818	3.591	3.421	3.288	3.182	3.094
27	7.677	5.488	4.601	4.106	3.785	3.558	3.388	3.256	3.149	3.062
28	7.636	5.453	4.568	4.074	3.754	3.528	3.358	3.226	3.120	3.032
29	7.598	5.420	4.538	4.045	3.725	3.499	3.330	3.198	3.092	3.005
30	7.562	5.390	4.510	4.018	3.699	3.473	3.305	3.173	3.067	2.979
31	7.530	5.362	4.484	3.993	3.675	3.449	3.281	3.149	3.043	2.955
32	7.499	5.336	4.459	3.969	3.652	3.427	3.258	3.127	3.021	2.934
33	7.471	5.312	4.437	3.948	3.630	3.406	3.238	3.106	3.000	2.913
34	7.444	5.289	4.416	3.927	3.611	3.386	3.218	3.087	2.981	2.894
35	7.419	5.268	4.396	3.908	3.592	3.368	3.200	3.069	2.963	2.876
36	7.396	5.248	4.377	3.890	3.574	3.351	3.183	3.052	2.946	2.859
37	7.373	5.229	4.360	3.873	3.558	3.334	3.167	3.036	2.930	2.843
38	7.353	5.211	4.343	3.858	3.542	3.319	3.152	3.021	2.915	2.828
39	7.333	5.194	4.327	3.843	3.528	3.305	3.137	3.006	2.901	2.814
40	7.314	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.888	2.801
41	7.296	5.163	4.299	3.815	3.501	3.278	3.111	2.980	2.875	2.788

42	7.280	5.149	4.285	3.802	3.488	3.266	3.099	2.968	2.863	2.776
43	7.264	5.136	4.273	3.790	3.476	3.254	3.087	2.957	2.851	2.764
44	7.248	5.123	4.261	3.778	3.465	3.243	3.076	2.946	2.840	2.754
45	7.234	5.110	4.249	3.767	3.454	3.232	3.066	2.935	2.830	2.743
46	7.220	5.099	4.238	3.757	3.444	3.222	3.056	2.925	2.820	2.733
47	7.207	5.087	4.228	3.747	3.434	3.213	3.046	2.916	2.811	2.724
48	7.194	5.077	4.218	3.737	3.425	3.204	3.037	2.907	2.802	2.715
49	7.182	5.066	4.208	3.728	3.416	3.195	3.028	2.898	2.793	2.706
50	7.171	5.057	4.199	3.720	3.408	3.186	3.020	2.890	2.785	2.698
51	7.159	5.047	4.191	3.711	3.400	3.178	3.012	2.882	2.777	2.690
52	7.149	5.038	4.182	3.703	3.392	3.171	3.005	2.874	2.769	2.683
53	7.139	5.030	4.174	3.695	3.384	3.163	2.997	2.867	2.762	2.675
54	7.129	5.021	4.167	3.688	3.377	3.156	2.990	2.860	2.755	2.668
55	7.119	5.013	4.159	3.681	3.370	3.149	2.983	2.853	2.748	2.662
56	7.110	5.006	4.152	3.674	3.363	3.143	2.977	2.847	2.742	2.655
57	7.102	4.998	4.145	3.667	3.357	3.136	2.971	2.841	2.736	2.649
58	7.093	4.991	4.138	3.661	3.351	3.130	2.965	2.835	2.730	2.643
59	7.085	4.984	4.132	3.655	3.345	3.124	2.959	2.829	2.724	2.637
60	7.077	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.718	2.632
61	7.070	4.971	4.120	3.643	3.333	3.113	2.948	2.818	2.713	2.626
62	7.062	4.965	4.114	3.638	3.328	3.108	2.942	2.813	2.708	2.621
63	7.055	4.959	4.109	3.632	3.323	3.103	2.937	2.808	2.703	2.616
64	7.048	4.953	4.103	3.627	3.318	3.098	2.932	2.803	2.698	2.611
65	7.042	4.947	4.098	3.622	3.313	3.093	2.928	2.798	2.693	2.607
66	7.035	4.942	4.093	3.618	3.308	3.088	2.923	2.793	2.689	2.602
67	7.029	4.937	4.088	3.613	3.304	3.084	2.919	2.789	2.684	2.598
68	7.023	4.932	4.083	3.608	3.299	3.080	2.914	2.785	2.680	2.593
69	7.017	4.927	4.079	3.604	3.295	3.075	2.910	2.781	2.676	2.589
70	7.011	4.922	4.074	3.600	3.291	3.071	2.906	2.777	2.672	2.585
71	7.006	4.917	4.070	3.596	3.287	3.067	2.902	2.773	2.668	2.581
72	7.001	4.913	4.066	3.591	3.283	3.063	2.898	2.769	2.664	2.578
73	6.995	4.908	4.062	3.588	3.279	3.060	2.895	2.765	2.660	2.574

74	6.990	4.904	4.058	3.584	3.275	3.056	2.891	2.762	2.657	2.570
75	6.985	4.900	4.054	3.580	3.272	3.052	2.887	2.758	2.653	2.567
76	6.981	4.896	4.050	3.577	3.268	3.049	2.884	2.755	2.650	2.563
77	6.976	4.892	4.047	3.573	3.265	3.046	2.881	2.751	2.647	2.560
78	6.971	4.888	4.043	3.570	3.261	3.042	2.877	2.748	2.644	2.557
79	6.967	4.884	4.040	3.566	3.258	3.039	2.874	2.745	2.640	2.554
80	6.963	4.881	4.036	3.563	3.255	3.036	2.871	2.742	2.637	2.551
81	6.958	4.877	4.033	3.560	3.252	3.033	2.868	2.739	2.634	2.548
82	6.954	4.874	4.030	3.557	3.249	3.030	2.865	2.736	2.632	2.545
83	6.950	4.870	4.027	3.554	3.246	3.027	2.863	2.733	2.629	2.542
84	6.947	4.867	4.024	3.551	3.243	3.025	2.860	2.731	2.626	2.539
85	6.943	4.864	4.021	3.548	3.240	3.022	2.857	2.728	2.623	2.537
86	6.939	4.861	4.018	3.545	3.238	3.019	2.854	2.725	2.621	2.534
87	6.935	4.858	4.015	3.543	3.235	3.017	2.852	2.723	2.618	2.532
88	6.932	4.855	4.012	3.540	3.233	3.014	2.849	2.720	2.616	2.529
89	6.928	4.852	4.010	3.538	3.230	3.012	2.847	2.718	2.613	2.527
90	6.925	4.849	4.007	3.535	3.228	3.009	2.845	2.715	2.611	2.524
91	6.922	4.846	4.004	3.533	3.225	3.007	2.842	2.713	2.609	2.522
92	6.919	4.844	4.002	3.530	3.223	3.004	2.840	2.711	2.606	2.520
93	6.915	4.841	3.999	3.528	3.221	3.002	2.838	2.709	2.604	2.518
94	6.912	4.838	3.997	3.525	3.218	3.000	2.835	2.706	2.602	2.515
95	6.909	4.836	3.995	3.523	3.216	2.998	2.833	2.704	2.600	2.513
96	6.906	4.833	3.992	3.521	3.214	2.996	2.831	2.702	2.598	2.511
97	6.904	4.831	3.990	3.519	3.212	2.994	2.829	2.700	2.596	2.509
98	6.901	4.829	3.988	3.517	3.210	2.992	2.827	2.698	2.594	2.507
99	6.898	4.826	3.986	3.515	3.208	2.990	2.825	2.696	2.592	2.505
100	6.895	4.824	3.984	3.513	3.206	2.988	2.823	2.694	2.590	2.503

\ ν_1 ν_2	11	12	13	14	15	16	17	18	19	20
1.	6083.35	6106.35	6125.86	6142.70	6157.28	6170.12	6181.42	6191.52	6200.58	6208.74

2.	99.408	99.416	99.422	99.428	99.432	99.437	99.440	99.444	99.447	99.449
3.	27.133	27.052	26.983	26.924	26.872	26.827	26.787	26.751	26.719	26.690
4.	14.452	14.374	14.307	14.249	14.198	14.154	14.115	14.080	14.048	14.020
5.	9.963	9.888	9.825	9.770	9.722	9.680	9.643	9.610	9.580	9.553
6.	7.790	7.718	7.657	7.605	7.559	7.519	7.483	7.451	7.422	7.396
7.	6.538	6.469	6.410	6.359	6.314	6.275	6.240	6.209	6.181	6.155
8.	5.734	5.667	5.609	5.559	5.515	5.477	5.442	5.412	5.384	5.359
9.	5.178	5.111	5.055	5.005	4.962	4.924	4.890	4.860	4.833	4.808
10.	4.772	4.706	4.650	4.601	4.558	4.520	4.487	4.457	4.430	4.405
11.	4.462	4.397	4.342	4.293	4.251	4.213	4.180	4.150	4.123	4.099
12.	4.220	4.155	4.100	4.052	4.010	3.972	3.939	3.909	3.883	3.858
13.	4.025	3.960	3.905	3.857	3.815	3.778	3.745	3.716	3.689	3.665
14.	3.864	3.800	3.745	3.698	3.656	3.619	3.586	3.556	3.529	3.505
15.	3.730	3.666	3.612	3.564	3.522	3.485	3.452	3.423	3.396	3.372
16.	3.616	3.553	3.498	3.451	3.409	3.372	3.339	3.310	3.283	3.259
17.	3.519	3.455	3.401	3.353	3.312	3.275	3.242	3.212	3.186	3.162
18.	3.434	3.371	3.316	3.269	3.227	3.190	3.158	3.128	3.101	3.077
19.	3.360	3.297	3.242	3.195	3.153	3.116	3.084	3.054	3.027	3.003
20.	3.294	3.231	3.177	3.130	3.088	3.051	3.018	2.989	2.962	2.938
21.	3.236	3.173	3.119	3.072	3.030	2.993	2.960	2.931	2.904	2.880
22.	3.184	3.121	3.067	3.019	2.978	2.941	2.908	2.879	2.852	2.827
23.	3.137	3.074	3.020	2.973	2.931	2.894	2.861	2.832	2.805	2.781
24.	3.094	3.032	2.977	2.930	2.889	2.852	2.819	2.789	2.762	2.738
25.	3.056	2.993	2.939	2.892	2.850	2.813	2.780	2.751	2.724	2.699
26.	3.021	2.958	2.904	2.857	2.815	2.778	2.745	2.715	2.688	2.664
27.	2.988	2.926	2.871	2.824	2.783	2.746	2.713	2.683	2.656	2.632
28.	2.959	2.896	2.842	2.795	2.753	2.716	2.683	2.653	2.626	2.602
29.	2.931	2.868	2.814	2.767	2.726	2.689	2.656	2.626	2.599	2.574
30.	2.906	2.843	2.789	2.742	2.700	2.663	2.630	2.600	2.573	2.549
31.	2.882	2.820	2.765	2.718	2.677	2.640	2.606	2.577	2.550	2.525
32.	2.860	2.798	2.744	2.696	2.655	2.618	2.584	2.555	2.527	2.503
33.	2.840	2.777	2.723	2.676	2.634	2.597	2.564	2.534	2.507	2.482

34.	2.821	2.758	2.704	2.657	2.615	2.578	2.545	2.515	2.488	2.463
35.	2.803	2.740	2.686	2.639	2.597	2.560	2.527	2.497	2.470	2.445
36.	2.786	2.723	2.669	2.622	2.580	2.543	2.510	2.480	2.453	2.428
37.	2.770	2.707	2.653	2.606	2.564	2.527	2.494	2.464	2.437	2.412
38.	2.755	2.692	2.638	2.591	2.549	2.512	2.479	2.449	2.421	2.397
39.	2.741	2.678	2.624	2.577	2.535	2.498	2.465	2.434	2.407	2.382
40.	2.727	2.665	2.611	2.563	2.522	2.484	2.451	2.421	2.394	2.369
41.	2.715	2.652	2.598	2.551	2.509	2.472	2.438	2.408	2.381	2.356
42.	2.703	2.640	2.586	2.539	2.497	2.460	2.426	2.396	2.369	2.344
43.	2.691	2.629	2.575	2.527	2.485	2.448	2.415	2.385	2.357	2.332
44.	2.680	2.618	2.564	2.516	2.475	2.437	2.404	2.374	2.346	2.321
45.	2.670	2.608	2.553	2.506	2.464	2.427	2.393	2.363	2.336	2.311
46.	2.660	2.598	2.544	2.496	2.454	2.417	2.384	2.353	2.326	2.301
47.	2.651	2.588	2.534	2.487	2.445	2.408	2.374	2.344	2.316	2.291
48.	2.642	2.579	2.525	2.478	2.436	2.399	2.365	2.335	2.307	2.282
49.	2.633	2.571	2.517	2.469	2.427	2.390	2.356	2.326	2.299	2.274
50.	2.625	2.562	2.508	2.461	2.419	2.382	2.348	2.318	2.290	2.265
51.	2.617	2.555	2.500	2.453	2.411	2.374	2.340	2.310	2.282	2.257
52.	2.610	2.547	2.493	2.445	2.403	2.366	2.333	2.302	2.275	2.250
53.	2.602	2.540	2.486	2.438	2.396	2.359	2.325	2.295	2.267	2.242
54.	2.595	2.533	2.479	2.431	2.389	2.352	2.318	2.288	2.260	2.235
55.	2.589	2.526	2.472	2.424	2.382	2.345	2.311	2.281	2.253	2.228
56.	2.582	2.520	2.465	2.418	2.376	2.339	2.305	2.275	2.247	2.222
57.	2.576	2.513	2.459	2.412	2.370	2.332	2.299	2.268	2.241	2.215
58.	2.570	2.507	2.453	2.406	2.364	2.326	2.293	2.262	2.235	2.209
59.	2.564	2.502	2.447	2.400	2.358	2.320	2.287	2.256	2.229	2.203
60.	2.559	2.496	2.442	2.394	2.352	2.315	2.281	2.251	2.223	2.198
61.	2.553	2.491	2.436	2.389	2.347	2.309	2.276	2.245	2.218	2.192
62.	2.548	2.486	2.431	2.384	2.342	2.304	2.270	2.240	2.212	2.187
63.	2.543	2.481	2.426	2.379	2.337	2.299	2.265	2.235	2.207	2.182
64.	2.538	2.476	2.421	2.374	2.332	2.294	2.260	2.230	2.202	2.177
65.	2.534	2.471	2.417	2.369	2.327	2.289	2.256	2.225	2.198	2.172

66.	2.529	2.466	2.412	2.365	2.322	2.285	2.251	2.221	2.193	2.168
67.	2.525	2.462	2.408	2.360	2.318	2.280	2.247	2.216	2.188	2.163
68.	2.520	2.458	2.403	2.356	2.314	2.276	2.242	2.212	2.184	2.159
69.	2.516	2.454	2.399	2.352	2.310	2.272	2.238	2.208	2.180	2.155
70.	2.512	2.450	2.395	2.348	2.306	2.268	2.234	2.204	2.176	2.150
71.	2.508	2.446	2.391	2.344	2.302	2.264	2.230	2.200	2.172	2.146
72.	2.504	2.442	2.388	2.340	2.298	2.260	2.226	2.196	2.168	2.143
73.	2.501	2.438	2.384	2.336	2.294	2.256	2.223	2.192	2.164	2.139
74.	2.497	2.435	2.380	2.333	2.290	2.253	2.219	2.188	2.161	2.135
75.	2.494	2.431	2.377	2.329	2.287	2.249	2.215	2.185	2.157	2.132
76.	2.490	2.428	2.373	2.326	2.284	2.246	2.212	2.181	2.154	2.128
77.	2.487	2.424	2.370	2.322	2.280	2.243	2.209	2.178	2.150	2.125
78.	2.484	2.421	2.367	2.319	2.277	2.239	2.206	2.175	2.147	2.122
79.	2.481	2.418	2.364	2.316	2.274	2.236	2.202	2.172	2.144	2.118
80.	2.478	2.415	2.361	2.313	2.271	2.233	2.199	2.169	2.141	2.115
81.	2.475	2.412	2.358	2.310	2.268	2.230	2.196	2.166	2.138	2.112
82.	2.472	2.409	2.355	2.307	2.265	2.227	2.193	2.163	2.135	2.109
83.	2.469	2.406	2.352	2.304	2.262	2.224	2.191	2.160	2.132	2.106
84.	2.466	2.404	2.349	2.302	2.259	2.222	2.188	2.157	2.129	2.104
85.	2.464	2.401	2.347	2.299	2.257	2.219	2.185	2.154	2.126	2.101
86.	2.461	2.398	2.344	2.296	2.254	2.216	2.182	2.152	2.124	2.098
87.	2.459	2.396	2.342	2.294	2.252	2.214	2.180	2.149	2.121	2.096
88.	2.456	2.393	2.339	2.291	2.249	2.211	2.177	2.147	2.119	2.093
89.	2.454	2.391	2.337	2.289	2.247	2.209	2.175	2.144	2.116	2.091
90.	2.451	2.389	2.334	2.286	2.244	2.206	2.172	2.142	2.114	2.088
91.	2.449	2.386	2.332	2.284	2.242	2.204	2.170	2.139	2.111	2.086
92.	2.447	2.384	2.330	2.282	2.240	2.202	2.168	2.137	2.109	2.083
93.	2.444	2.382	2.327	2.280	2.237	2.200	2.166	2.135	2.107	2.081
94.	2.442	2.380	2.325	2.277	2.235	2.197	2.163	2.133	2.105	2.079
95.	2.440	2.378	2.323	2.275	2.233	2.195	2.161	2.130	2.102	2.077
96.	2.438	2.375	2.321	2.273	2.231	2.193	2.159	2.128	2.100	2.075
97.	2.436	2.373	2.319	2.271	2.229	2.191	2.157	2.126	2.098	2.073

```
  98.        2.434    2.371    2.317    2.269    2.227
 2.189    2.155    2.124    2.096    2.071
  99.        2.432    2.369    2.315    2.267    2.225
 2.187    2.153    2.122    2.094    2.069
 100.        2.430    2.368    2.313    2.265    2.223
 2.185    2.151    2.120    2.092    2.067
```

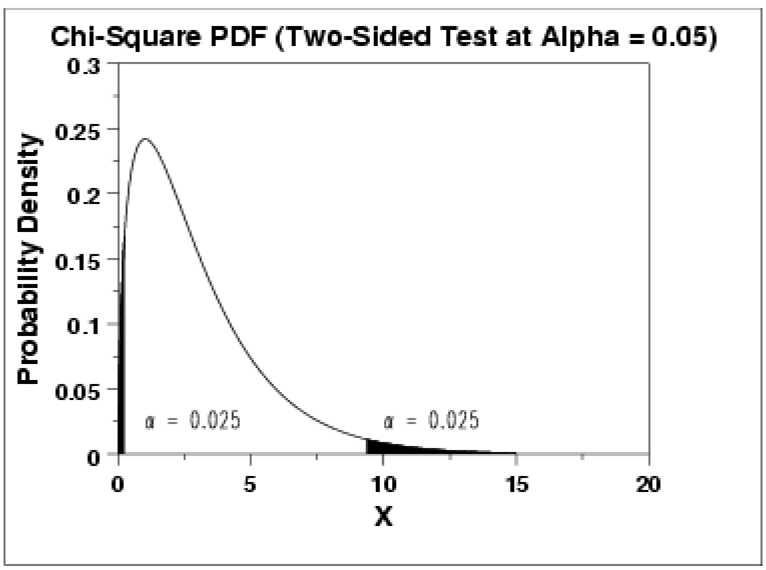
1.3.6.7.4. Critical Values of the Chi-Square Distribution

How to Use This Table

This table contains the critical values of the chi-square distribution. Because of the lack of symmetry of the chi-square distribution, separate tables are provided for the upper and lower tails of the distribution.

A test statistic with ν degrees of freedom is computed from the data. For upper-tail one-sided tests, the test statistic is compared with a value from the table of upper-tail critical values. For two-sided tests, the test statistic is compared with values from both the table for the upper-tail critical values and the table for the lower-tail critical values.

The significance level, α, is demonstrated with the graph below which shows a chi-square distribution with 3 degrees of freedom for a two-sided test at significance level α = 0.05. If the test statistic is greater than the upper-tail critical value or less than the lower-tail critical value, we reject the null hypothesis. Specific instructions are given below.



Given a specified value of α:

1. For a two-sided test, find the column corresponding to 1-α/2 in the table for upper-tail critical values and reject

the null hypothesis if the test statistic is greater than the tabled value. Similarly, find the column corresponding to $\alpha/2$ in the table for lower-tail critical values and reject the null hypothesis if the test statistic is less than the tabled value.

2. For an upper-tail one-sided test, find the column corresponding to 1-α in the table containing upper-tail critical and reject the null hypothesis if the test statistic is greater than the tabled value.
3. For a lower-tail one-sided test, find the column corresponding to α in the lower-tail critical values table and reject the null hypothesis if the computed test statistic is less than the tabled value.

Upper-tail critical values of chi-square distribution with ν degrees of freedom

	Probability less than the critical value				
ν	0.90	0.95	0.975	0.99	0.999
1	2.706	3.841	5.024	6.635	10.828
2	4.605	5.991	7.378	9.210	13.816
3	6.251	7.815	9.348	11.345	16.266
4	7.779	9.488	11.143	13.277	18.467
5	9.236	11.070	12.833	15.086	20.515
6	10.645	12.592	14.449	16.812	22.458
7	12.017	14.067	16.013	18.475	24.322
8	13.362	15.507	17.535	20.090	26.125
9	14.684	16.919	19.023	21.666	27.877
10	15.987	18.307	20.483	23.209	29.588
11	17.275	19.675	21.920	24.725	31.264
12	18.549	21.026	23.337	26.217	32.910
13	19.812	22.362	24.736	27.688	34.528
14	21.064	23.685	26.119	29.141	36.123
15	22.307	24.996	27.488	30.578	37.697
16	23.542	26.296	28.845	32.000	39.252

17	24.769	27.587	30.191	33.409	40.790
18	25.989	28.869	31.526	34.805	42.312
19	27.204	30.144	32.852	36.191	43.820
20	28.412	31.410	34.170	37.566	45.315
21	29.615	32.671	35.479	38.932	46.797
22	30.813	33.924	36.781	40.289	48.268
23	32.007	35.172	38.076	41.638	49.728
24	33.196	36.415	39.364	42.980	51.179
25	34.382	37.652	40.646	44.314	52.620
26	35.563	38.885	41.923	45.642	54.052
27	36.741	40.113	43.195	46.963	55.476
28	37.916	41.337	44.461	48.278	56.892
29	39.087	42.557	45.722	49.588	58.301
30	40.256	43.773	46.979	50.892	59.703
31	41.422	44.985	48.232	52.191	61.098
32	42.585	46.194	49.480	53.486	62.487
33	43.745	47.400	50.725	54.776	63.870
34	44.903	48.602	51.966	56.061	65.247
35	46.059	49.802	53.203	57.342	66.619
36	47.212	50.998	54.437	58.619	67.985
37	48.363	52.192	55.668	59.893	69.347
38	49.513	53.384	56.896	61.162	70.703
39	50.660	54.572	58.120	62.428	72.055
40	51.805	55.758	59.342	63.691	73.402
41	52.949	56.942	60.561	64.950	74.745
42	54.090	58.124	61.777	66.206	76.084
43	55.230	59.304	62.990	67.459	77.419
44	56.369	60.481	64.201	68.710	78.750
45	57.505	61.656	65.410	69.957	80.077
46	58.641	62.830	66.617	71.201	81.400
47	59.774	64.001	67.821	72.443	82.720
48	60.907	65.171	69.023	73.683	84.037

49	62.038	66.339	70.222	74.919	85.351
50	63.167	67.505	71.420	76.154	86.661
51	64.295	68.669	72.616	77.386	87.968
52	65.422	69.832	73.810	78.616	89.272
53	66.548	70.993	75.002	79.843	90.573
54	67.673	72.153	76.192	81.069	91.872
55	68.796	73.311	77.380	82.292	93.168
56	69.919	74.468	78.567	83.513	94.461
57	71.040	75.624	79.752	84.733	95.751
58	72.160	76.778	80.936	85.950	97.039
59	73.279	77.931	82.117	87.166	98.324
60	74.397	79.082	83.298	88.379	99.607
61	75.514	80.232	84.476	89.591	100.888
62	76.630	81.381	85.654	90.802	102.166
63	77.745	82.529	86.830	92.010	103.442
64	78.860	83.675	88.004	93.217	104.716
65	79.973	84.821	89.177	94.422	105.988
66	81.085	85.965	90.349	95.626	107.258
67	82.197	87.108	91.519	96.828	108.526
68	83.308	88.250	92.689	98.028	109.791
69	84.418	89.391	93.856	99.228	111.055
70	85.527	90.531	95.023	100.425	112.317
71	86.635	91.670	96.189	101.621	113.577
72	87.743	92.808	97.353	102.816	114.835
73	88.850	93.945	98.516	104.010	116.092
74	89.956	95.081	99.678	105.202	117.346
75	91.061	96.217	100.839	106.393	118.599
76	92.166	97.351	101.999	107.583	119.850
77	93.270	98.484	103.158	108.771	121.100
78	94.374	99.617	104.316	109.958	122.348
79	95.476	100.749	105.473	111.144	123.594
80	96.578	101.879	106.629	112.329	124.839

81	97.680	103.010	107.783	113.512	126.083
82	98.780	104.139	108.937	114.695	127.324
83	99.880	105.267	110.090	115.876	128.565
84	100.980	106.395	111.242	117.057	129.804
85	102.079	107.522	112.393	118.236	131.041
86	103.177	108.648	113.544	119.414	132.277
87	104.275	109.773	114.693	120.591	133.512
88	105.372	110.898	115.841	121.767	134.746
89	106.469	112.022	116.989	122.942	135.978
90	107.565	113.145	118.136	124.116	137.208
91	108.661	114.268	119.282	125.289	138.438
92	109.756	115.390	120.427	126.462	139.666
93	110.850	116.511	121.571	127.633	140.893
94	111.944	117.632	122.715	128.803	142.119
95	113.038	118.752	123.858	129.973	143.344
96	114.131	119.871	125.000	131.141	144.567
97	115.223	120.990	126.141	132.309	145.789
98	116.315	122.108	127.282	133.476	147.010
99	117.407	123.225	128.422	134.642	148.230
100	118.498	124.342	129.561	135.807	149.449
100	118.498	124.342	129.561	135.807	149.449

Lower-tail critical values of chi-square distribution with ν degrees of freedom

	Probability less than the critical value				
ν	0.10	0.05	0.025	0.01	0.001
1.	.016	.004	.001	.000	.000
2.	.211	.103	.051	.020	.002

3.	.584	.352	.216	.115	.024
4.	1.064	.711	.484	.297	.091
5.	1.610	1.145	.831	.554	.210
6.	2.204	1.635	1.237	.872	.381
7.	2.833	2.167	1.690	1.239	.598
8.	3.490	2.733	2.180	1.646	.857
9.	4.168	3.325	2.700	2.088	1.152
10.	4.865	3.940	3.247	2.558	1.479
11.	5.578	4.575	3.816	3.053	1.834
12.	6.304	5.226	4.404	3.571	2.214
13.	7.042	5.892	5.009	4.107	2.617
14.	7.790	6.571	5.629	4.660	3.041
15.	8.547	7.261	6.262	5.229	3.483
16.	9.312	7.962	6.908	5.812	3.942
17.	10.085	8.672	7.564	6.408	4.416
18.	10.865	9.390	8.231	7.015	4.905
19.	11.651	10.117	8.907	7.633	5.407
20.	12.443	10.851	9.591	8.260	5.921
21.	13.240	11.591	10.283	8.897	6.447
22.	14.041	12.338	10.982	9.542	6.983
23.	14.848	13.091	11.689	10.196	7.529
24.	15.659	13.848	12.401	10.856	8.085
25.	16.473	14.611	13.120	11.524	8.649
26.	17.292	15.379	13.844	12.198	9.222
27.	18.114	16.151	14.573	12.879	9.803
28.	18.939	16.928	15.308	13.565	10.391
29.	19.768	17.708	16.047	14.256	10.986
30.	20.599	18.493	16.791	14.953	11.588
31.	21.434	19.281	17.539	15.655	12.196
32.	22.271	20.072	18.291	16.362	12.811
33.	23.110	20.867	19.047	17.074	13.431
34.	23.952	21.664	19.806	17.789	14.057

35.	24.797	22.465	20.569	18.509	14.688
36.	25.643	23.269	21.336	19.233	15.324
37.	26.492	24.075	22.106	19.960	15.965
38.	27.343	24.884	22.878	20.691	16.611
39.	28.196	25.695	23.654	21.426	17.262
40.	29.051	26.509	24.433	22.164	17.916
41.	29.907	27.326	25.215	22.906	18.575
42.	30.765	28.144	25.999	23.650	19.239
43.	31.625	28.965	26.785	24.398	19.906
44.	32.487	29.787	27.575	25.148	20.576
45.	33.350	30.612	28.366	25.901	21.251
46.	34.215	31.439	29.160	26.657	21.929
47.	35.081	32.268	29.956	27.416	22.610
48.	35.949	33.098	30.755	28.177	23.295
49.	36.818	33.930	31.555	28.941	23.983
50.	37.689	34.764	32.357	29.707	24.674
51.	38.560	35.600	33.162	30.475	25.368
52.	39.433	36.437	33.968	31.246	26.065
53.	40.308	37.276	34.776	32.018	26.765
54.	41.183	38.116	35.586	32.793	27.468
55.	42.060	38.958	36.398	33.570	28.173
56.	42.937	39.801	37.212	34.350	28.881
57.	43.816	40.646	38.027	35.131	29.592
58.	44.696	41.492	38.844	35.913	30.305
59.	45.577	42.339	39.662	36.698	31.020
60.	46.459	43.188	40.482	37.485	31.738
61.	47.342	44.038	41.303	38.273	32.459
62.	48.226	44.889	42.126	39.063	33.181
63.	49.111	45.741	42.950	39.855	33.906
64.	49.996	46.595	43.776	40.649	34.633
65.	50.883	47.450	44.603	41.444	35.362
66.	51.770	48.305	45.431	42.240	36.093

67.	52.659	49.162	46.261	43.038	36.826
68.	53.548	50.020	47.092	43.838	37.561
69.	54.438	50.879	47.924	44.639	38.298
70.	55.329	51.739	48.758	45.442	39.036
71.	56.221	52.600	49.592	46.246	39.777
72.	57.113	53.462	50.428	47.051	40.519
73.	58.006	54.325	51.265	47.858	41.264
74.	58.900	55.189	52.103	48.666	42.010
75.	59.795	56.054	52.942	49.475	42.757
76.	60.690	56.920	53.782	50.286	43.507
77.	61.586	57.786	54.623	51.097	44.258
78.	62.483	58.654	55.466	51.910	45.010
79.	63.380	59.522	56.309	52.725	45.764
80.	64.278	60.391	57.153	53.540	46.520
81.	65.176	61.261	57.998	54.357	47.277
82.	66.076	62.132	58.845	55.174	48.036
83.	66.976	63.004	59.692	55.993	48.796
84.	67.876	63.876	60.540	56.813	49.557
85.	68.777	64.749	61.389	57.634	50.320
86.	69.679	65.623	62.239	58.456	51.085
87.	70.581	66.498	63.089	59.279	51.850
88.	71.484	67.373	63.941	60.103	52.617
89.	72.387	68.249	64.793	60.928	53.386
90.	73.291	69.126	65.647	61.754	54.155
91.	74.196	70.003	66.501	62.581	54.926
92.	75.100	70.882	67.356	63.409	55.698
93.	76.006	71.760	68.211	64.238	56.472
94.	76.912	72.640	69.068	65.068	57.246
95.	77.818	73.520	69.925	65.898	58.022
96.	78.725	74.401	70.783	66.730	58.799
97.	79.633	75.282	71.642	67.562	59.577
98.	80.541	76.164	72.501	68.396	60.356

99.	81.449	77.046	73.361	69.230	61.137
100.	82.358	77.929	74.222	70.065	61.918

1. Exploratory Data Analysis
1.3. EDA Techniques
1.3.6. Probability Distributions
1.3.6.7. Tables for Probability Distributions

1.3.6.7.5. Critical Values of the t^* Distribution

How to Use This Table

This table contains upper critical values of the t* distribution that are appropriate for determining whether or not a calibration line is in a state of statistical control from measurements on a check standard at three points in the calibration interval. A test statistic with ν degrees of freedom is compared with the critical value. If the absolute value of the test statistic exceeds the tabled value, the calibration of the instrument is judged to be out of control.

Upper critical values of t* distribution at significance level 0.05 for testing the output of a linear calibration line at 3 points

ν	$t^*_{.05}(\nu)$	ν	$t^*_{.05}(\nu)$
1	37.544	61	2.455
2	7.582	62	2.454
3	4.826	63	2.453
4	3.941	64	2.452
5	3.518	65	2.451
6	3.274	66	2.450
7	3.115	67	2.449
8	3.004	68	2.448
9	2.923	69	2.447
10	2.860	70	2.446
11	2.811	71	2.445
12	2.770	72	2.445
13	2.737	73	2.444
14	2.709	74	2.443
15	2.685	75	2.442
16	2.665	76	2.441
17	2.647	77	2.441
18	2.631	78	2.440
19	2.617	79	2.439
20	2.605	80	2.439
21	2.594	81	2.438
22	2.584	82	2.437
23	2.574	83	2.437
24	2.566	84	2.436
25	2.558	85	2.436
26	2.551	86	2.435
27	2.545	87	2.435
28	2.539	88	2.434
29	2.534	89	2.434
30	2.528	90	2.433
31	2.524	91	2.432
32	2.519	92	2.432
33	2.515	93	2.431
34	2.511	94	2.431
35	2.507	95	2.431

36	2.504	96	2.430
37	2.501	97	2.430
38	2.498	98	2.429
39	2.495	99	2.429
40	2.492	100	2.428
41	2.489	101	2.428
42	2.487	102	2.428
43	2.484	103	2.427
44	2.482	104	2.427
45	2.480	105	2.426
46	2.478	106	2.426
47	2.476	107	2.426
48	2.474	108	2.425
49	2.472	109	2.425
50	2.470	110	2.425
51	2.469	111	2.424
52	2.467	112	2.424
53	2.466	113	2.424
54	2.464	114	2.423
55	2.463	115	2.423
56	2.461	116	2.423
57	2.460	117	2.422
58	2.459	118	2.422
59	2.457	119	2.422
60	2.456	120	2.422

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.3.6.7.6. Critical Values of the Normal PPCC Distribution

How to Use This Table

This table contains the critical values of the normal probability plot correlation coefficient (PPCC) distribution that are appropriate for determining whether or not a data set came from a population with approximately a normal distribution. It is used in conjuction with a normal probability plot. The test statistic is the correlation coefficient of the points that make up a normal probability plot. This test statistic is compared with the critical value below. If the test statistic is less than the tabulated value, the null hypothesis that the data came from a population with a normal distribution is rejected.

For example, suppose a set of 50 data points had a correlation coefficient of 0.985 from the normal probability plot. At the 5% significance level, the critical value is 0.9761. Since 0.985 is greater than 0.9761, we cannot reject the null hypothesis that the data came from a population with a normal distribution.

Since perferct normality implies perfect correlation (i.e., a correlation value of 1), we are only interested in rejecting normality for correlation values that are too low. That is, this is a lower one-tailed test.

The values in this table were determined from simulation studies by Filliben and Devaney.

Critical values of the normal PPCC for testing if data come from a normal distribution

N	0.01	0.05
3	0.8687	0.8790
4	0.8234	0.8666
5	0.8240	0.8786
6	0.8351	0.8880
7	0.8474	0.8970
8	0.8590	0.9043
9	0.8689	0.9115
10	0.8765	0.9173
11	0.8838	0.9223
12	0.8918	0.9267
13	0.8974	0.9310
14	0.9029	0.9343

15	0.9080	0.9376
16	0.9121	0.9405
17	0.9160	0.9433
18	0.9196	0.9452
19	0.9230	0.9479
20	0.9256	0.9498
21	0.9285	0.9515
22	0.9308	0.9535
23	0.9334	0.9548
24	0.9356	0.9564
25	0.9370	0.9575
26	0.9393	0.9590
27	0.9413	0.9600
28	0.9428	0.9615
29	0.9441	0.9622
30	0.9462	0.9634
31	0.9476	0.9644
32	0.9490	0.9652
33	0.9505	0.9661
34	0.9521	0.9671
35	0.9530	0.9678
36	0.9540	0.9686
37	0.9551	0.9693
38	0.9555	0.9700
39	0.9568	0.9704
40	0.9576	0.9712
41	0.9589	0.9719
42	0.9593	0.9723
43	0.9609	0.9730
44	0.9611	0.9734
45	0.9620	0.9739
46	0.9629	0.9744
47	0.9637	0.9748
48	0.9640	0.9753
49	0.9643	0.9758
50	0.9654	0.9761
55	0.9683	0.9781
60	0.9706	0.9797
65	0.9723	0.9809
70	0.9742	0.9822
75	0.9758	0.9831
80	0.9771	0.9841
85	0.9784	0.9850
90	0.9797	0.9857
95	0.9804	0.9864
100	0.9814	0.9869
110	0.9830	0.9881
120	0.9841	0.9889
130	0.9854	0.9897
140	0.9865	0.9904
150	0.9871	0.9909
160	0.9879	0.9915
170	0.9887	0.9919
180	0.9891	0.9923
190	0.9897	0.9927
200	0.9903	0.9930
210	0.9907	0.9933
220	0.9910	0.9936
230	0.9914	0.9939
240	0.9917	0.9941
250	0.9921	0.9943
260	0.9924	0.9945
270	0.9926	0.9947
280	0.9929	0.9949
290	0.9931	0.9951
300	0.9933	0.9952
310	0.9936	0.9954
320	0.9937	0.9955
330	0.9939	0.9956
340	0.9941	0.9957
350	0.9942	0.9958
360	0.9944	0.9959
370	0.9945	0.9960
380	0.9947	0.9961

390	0.9948	0.9962
400	0.9949	0.9963
410	0.9950	0.9964
420	0.9951	0.9965
430	0.9953	0.9966
440	0.9954	0.9966
450	0.9954	0.9967
460	0.9955	0.9968
470	0.9956	0.9968
480	0.9957	0.9969
490	0.9958	0.9969
500	0.9959	0.9970
525	0.9961	0.9972
550	0.9963	0.9973
575	0.9964	0.9974
600	0.9965	0.9975
625	0.9967	0.9976
650	0.9968	0.9977
675	0.9969	0.9977
700	0.9970	0.9978
725	0.9971	0.9979
750	0.9972	0.9980
775	0.9973	0.9980
800	0.9974	0.9981
825	0.9975	0.9981
850	0.9975	0.9982
875	0.9976	0.9982
900	0.9977	0.9983
925	0.9977	0.9983
950	0.9978	0.9984
975	0.9978	0.9984
1000	0.9979	0.9984

HOME TOOLS & AIDS SEARCH BACK NEXT

1.4. EDA Case Studies

Summary

This section presents a series of case studies that demonstrate the application of EDA methods to specific problems. In some cases, we have focused on just one EDA technique that uncovers virtually all there is to know about the data. For other case studies, we need several EDA techniques, the selection of which is dictated by the outcome of the previous step in the analaysis sequence. Note in these case studies how the flow of the analysis is motivated by the focus on underlying assumptions and general EDA principles.

Table of Contents for Section 4

1.4.1. Case Studies Introduction

Purpose

The purpose of the first eight case studies is to show how EDA graphics and quantitative measures and tests are applied to data from scientific processes and to critique those data with regard to the following assumptions that typically underlie a measurement process; namely, that the data behave like:

- random drawings
- from a fixed distribution
- with a fixed location
- with a fixed standard deviation

Case studies 9 and 10 show the use of EDA techniques in distributional modeling and the analysis of a designed experiment, respectively.

$Y_i = C + E_i$

If the above assumptions are satisfied, the process is said to be statistically "in control" with the core characteristic of having "predictability". That is, probability statements can be made about the process, not only in the past, but also in the future.

An appropriate model for an "in control" process is

$$\boldsymbol{Y_i = C + E_i}$$

where $\boldsymbol{C}$ is a constant (the "deterministic" or "structural" component), and where $\boldsymbol{E_i}$ is the error term (or "random" component).

The constant $\boldsymbol{C}$ is the average value of the process--it is the primary summary number which shows up on any report. Although $\boldsymbol{C}$ is (assumed) fixed, it is unknown, and so a primary analysis objective of the engineer is to arrive at an estimate of $\boldsymbol{C}$.

This goal partitions into 4 sub-goals:

1. Is the most common estimator of $\boldsymbol{C}$, $\bar{Y}$, the best estimator for $\boldsymbol{C}$? What does "best" mean?
2. If $\bar{Y}$ is best, what is the uncertainty $s_{\bar{Y}}$ for $\bar{Y}$. In particular, is the usual formula for the uncertainty of

$\bar{Y}$:

$$s_{\bar{Y}} = s/\sqrt{N}$$

valid? Here, ***s*** is the standard deviation of the data and ***N*** is the sample size.

3. If $\bar{Y}$ is **not** the best estimator for ***C***, what is a better estimator for ***C*** (for example, median, midrange, midmean)?

4. If there is a better estimator, $\hat{C}$, what is its uncertainty? That is, what is $s_{\hat{C}}$?

EDA and the routine checking of underlying assumptions provides insight into all of the above.

1. Location and variation checks provide information as to whether ***C*** is really constant.

2. Distributional checks indicate whether $\bar{Y}$ is the best estimator. Techniques for distributional checking include histograms, normal probability plots, and probability plot correlation coefficient plots.

3. Randomness checks ascertain whether the usual

$$s_{\bar{Y}} = s/\sqrt{N}$$

is valid.

4. Distributional tests assist in determining a better estimator, if needed.

5. Simulator tools (namely bootstrapping) provide values for the uncertainty of alternative estimators.

Assumptions not satisfied

If one or more of the above assumptions is not satisfied, then we use EDA techniques, or some mix of EDA and classical techniques, to find a more appropriate model for the data. That is,

$$Y_i = D + E_i$$

where ***D*** is the deterministic part and ***E*** is an error component.

If the data are not random, then we may investigate fitting some simple time series models to the data. If the constant location and scale assumptions are violated, we may need to investigate the measurement process to see if there is an explanation.

The assumptions on the error term are still quite relevant in the sense that for an appropriate model the error component should follow the assumptions. The criterion for validating the model, or comparing competing models, is framed in terms of these assumptions.

Multivariable data

Although the case studies in this chapter utilize univariate data, the assumptions above are relevant for multivariable data as well.

If the data are not univariate, then we are trying to find a model

$$Y_i = F(X_1, ..., X_k) + E_i$$

where F is some function based on one or more variables. The error component, which is a univariate data set, of a good model should satisfy the assumptions given above. The criterion for validating and comparing models is based on how well the error component follows these assumptions.

The load cell calibration case study in the process modeling chapter shows an example of this in the regression context.

First three case studies utilize data with known characteristics

The first three case studies utilize data that are randomly generated from the following distributions:

- normal distribution with mean 0 and standard deviation 1
- uniform distribution with mean 0 and standard deviation $\sqrt{1/12}$ (uniform over the interval (0,1))
- random walk

The other univariate case studies utilize data from scientific processes. The goal is to determine if

$$Y_i = C + E_i$$

is a reasonable model. This is done by testing the underlying assumptions. If the assumptions are satisfied, then an estimate of C and an estimate of the uncertainty of C are computed. If the assumptions are not satisfied, we attempt to find a model where the error component does satisfy the underlying assumptions.

Graphical methods that are applied to the data

To test the underlying assumptions, each data set is analyzed using four graphical methods that are particularly suited for this purpose:

1. run sequence plot which is useful for detecting shifts of location or scale

2. lag plot which is useful for detecting non-randomness in the data

3. histogram which is useful for trying to determine the underlying distribution

4. normal probability plot for deciding whether the data follow the normal distribution

There are a number of other techniques for addressing the underlying assumptions. However, the four plots listed above provide an excellent opportunity for addressing all of the assumptions on a single page of graphics.

Additional graphical techniques are used in certain case studies to develop models that do have error components that satisfy the underlying assumptions.

Quantitative methods that are applied to the data

The normal and uniform random number data sets are also analyzed with the following quantitative techniques, which are explained in more detail in an earlier section:

1. Summary statistics which include:
   - mean
   - standard deviation
   - autocorrelation coefficient to test for randomness
   - normal and uniform probability plot correlation coefficients (ppcc) to test for a normal or uniform distribution, respectively
   - Wilk-Shapiro test for a normal distribution

2. Linear fit of the data as a function of time to assess drift (test for fixed location)

3. Bartlett test for fixed variance

4. Autocorrelation plot and coefficient to test for randomness

5. Runs test to test for lack of randomness

6. Anderson-Darling test for a normal distribution

7. Grubbs test for outliers

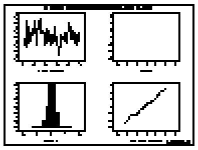
8. Summary report

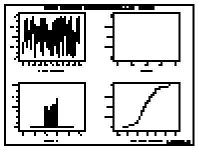
Although the graphical methods applied to the normal and uniform random numbers are sufficient to assess the validity of the underlying assumptions, the quantitative techniques are used to show the different flavor of the graphical and quantitative approaches.

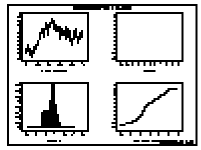
The remaining case studies intermix one or more of these quantitative techniques into the analysis where appropriate.

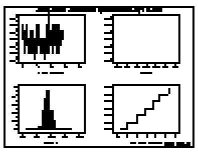
1.4.2. Case Studies

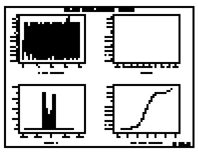
Univariate
$Y_i = C + E_i$

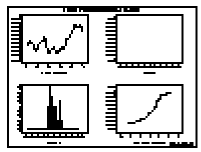

Normal Random Numbers

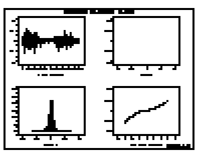

Uniform Random Numbers

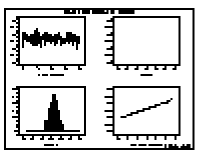

Random Walk


Josephson Junction Cryothermometry

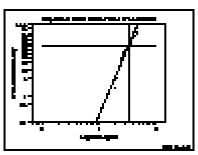

Beam Deflections


Filter Transmittance

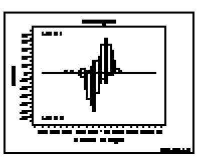

Standard Resistor


Heat Flow Meter 1

Reliability


Fatigue Life of Aluminum Alloy Specimens

Multi-Factor


Ceramic Strength

1.4.2.1. Normal Random Numbers

Normal Random Numbers

This example illustrates the univariate analysis of a set of normal random numbers.

1.4.2.1.1. Background and Data

Generation

The normal random numbers used in this case study are from a Rand Corporation publication.

The motivation for studying a set of normal random numbers is to illustrate the ideal case where all four underlying assumptions hold.

Software

The analyses used in this case study can be generated using both Dataplot code and R code.

Data

The following is the set of normal random numbers used for this case study.

```
 -1.2760 -1.2180 -0.4530 -0.3500  0.7230
  0.6760 -1.0990 -0.3140 -0.3940 -0.6330
 -0.3180 -0.7990 -1.6640  1.3910  0.3820
  0.7330  0.6530  0.2190 -0.6810  1.1290
 -1.3770 -1.2570  0.4950 -0.1390 -0.8540
  0.4280 -1.3220 -0.3150 -0.7320 -1.3480
  2.3340 -0.3370 -1.9550 -0.6360 -1.3180
 -0.4330  0.5450  0.4280 -0.2970  0.2760
 -1.1360  0.6420  3.4360 -1.6670  0.8470
 -1.1730 -0.3550  0.0350  0.3590  0.9300
  0.4140 -0.0110  0.6660 -1.1320 -0.4100
 -1.0770  0.7340  1.4840 -0.3400  0.7890
 -0.4940  0.3640 -1.2370 -0.0440 -0.1110
 -0.2100  0.9310  0.6160 -0.3770 -0.4330
  1.0480  0.0370  0.7590  0.6090 -2.0430
 -0.2900  0.4040 -0.5430  0.4860  0.8690
  0.3470  2.8160 -0.4640 -0.6320 -1.6140
  0.3720 -0.0740 -0.9160  1.3140 -0.0380
  0.6370  0.5630 -0.1070  0.1310 -1.8080
 -1.1260  0.3790  0.6100 -0.3640 -2.6260
  2.1760  0.3930 -0.9240  1.9110 -1.0400
 -1.1680  0.4850  0.0760 -0.7690  1.6070
 -1.1850 -0.9440 -1.6040  0.1850 -0.2580
 -0.3000 -0.5910 -0.5450  0.0180 -0.4850
  0.9720  1.7100  2.6820  2.8130 -1.5310
 -0.4900  2.0710  1.4440 -1.0920  0.4780
  1.2100  0.2940 -0.2480  0.7190  1.1030
  1.0900  0.2120 -1.1850 -0.3380 -1.1340
  2.6470  0.7770  0.4500  2.2470  1.1510
 -1.6760  0.3840  1.1330  1.3930  0.8140
  0.3980  0.3180 -0.9280  2.4160 -0.9360
  1.0360  0.0240 -0.5600  0.2030 -0.8710
  0.8460 -0.6990 -0.3680  0.3440 -0.9260
 -0.7970 -1.4040 -1.4720 -0.1180  1.4560
  0.6540 -0.9550  2.9070  1.6880  0.7520
 -0.4340  0.7460  0.1490 -0.1700 -0.4790
  0.5220  0.2310 -0.6190 -0.2650  0.4190
  0.5580 -0.5490  0.1920 -0.3340  1.3730
 -1.2880 -0.5390 -0.8240  0.2440 -1.0700
  0.0100  0.4820 -0.4690 -0.0900  1.1710
  1.3720  1.7690 -1.0570  1.6460  0.4810
 -0.6000 -0.5920  0.6100 -0.0960 -1.3750
```

0.8540	-0.5350	1.6070	0.4280	-0.6150
0.3310	-0.3360	-1.1520	0.5330	-0.8330
-0.1480	-1.1440	0.9130	0.6840	1.0430
0.5540	-0.0510	-0.9440	-0.4400	-0.2120
-1.1480	-1.0560	0.6350	-0.3280	-1.2210
0.1180	-2.0450	-1.9770	-1.1330	0.3380
0.3480	0.9700	-0.0170	1.2170	-0.9740
-1.2910	-0.3990	-1.2090	-0.2480	0.4800
0.2840	0.4580	1.3070	-1.6250	-0.6290
-0.5040	-0.0560	-0.1310	0.0480	1.8790
-1.0160	0.3600	-0.1190	2.3310	1.6720
-1.0530	0.8400	-0.2460	0.2370	-1.3120
1.6030	-0.9520	-0.5660	1.6000	0.4650
1.9510	0.1100	0.2510	0.1160	-0.9570
-0.1900	1.4790	-0.9860	1.2490	1.9340
0.0700	-1.3580	-1.2460	-0.9590	-1.2970
-0.7220	0.9250	0.7830	-0.4020	0.6190
1.8260	1.2720	-0.9450	0.4940	0.0500
-1.6960	1.8790	0.0630	0.1320	0.6820
0.5440	-0.4170	-0.6660	-0.1040	-0.2530
-2.5430	-1.3330	1.9870	0.6680	0.3600
1.9270	1.1830	1.2110	1.7650	0.3500
-0.3590	0.1930	-1.0230	-0.2220	-0.6160
-0.0600	-1.3190	0.7850	-0.4300	-0.2980
0.2480	-0.0880	-1.3790	0.2950	-0.1150
-0.6210	-0.6180	0.2090	0.9790	0.9060
-0.0990	-1.3760	1.0470	-0.8720	-2.2000
-1.3840	1.4250	-0.8120	0.7480	-1.0930
-0.4630	-1.2810	-2.5140	0.6750	1.1450
1.0830	-0.6670	-0.2230	-1.5920	-1.2780
0.5030	1.4340	0.2900	0.3970	-0.8370
-0.9730	-0.1200	-1.5940	-0.9960	-1.2440
-0.8570	-0.3710	-0.2160	0.1480	-2.1060
-1.4530	0.6860	-0.0750	-0.2430	-0.1700
-0.1220	1.1070	-1.0390	-0.6360	-0.8600
-0.8950	-1.4580	-0.5390	-0.1590	-0.4200
1.6320	0.5860	-0.4680	-0.3860	-0.3540
0.2030	-1.2340	2.3810	-0.3880	-0.0630
2.0720	-1.4450	-0.6800	0.2240	-0.1200
1.7530	-0.5710	1.2230	-0.1260	0.0340
-0.4350	-0.3750	-0.9850	-0.5850	-0.2030
-0.5560	0.0240	0.1260	1.2500	-0.6150
0.8760	-1.2270	-2.6470	-0.7450	1.7970
-1.2310	0.5470	-0.6340	-0.8360	-0.7190
0.8330	1.2890	-0.0220	-0.4310	0.5820
0.7660	-0.5740	-1.1530	0.5200	-1.0180
-0.8910	0.3320	-0.4530	-1.1270	2.0850
-0.7220	-1.5080	0.4890	-0.4960	-0.0250
0.6440	-0.2330	-0.1530	1.0980	0.7570
-0.0390	-0.4600	0.3930	2.0120	1.3560
0.1050	-0.1710	-0.1100	-1.1450	0.8780
-0.9090	-0.3280	1.0210	-1.6130	1.5600
-1.1920	1.7700	-0.0030	0.3690	0.0520
0.6470	1.0290	1.5260	0.2370	-1.3280
-0.0420	0.5530	0.7700	0.3240	-0.4890
-0.3670	0.3780	0.6010	-1.9960	-0.7380
0.4980	1.0720	1.5670	0.3020	1.1570
-0.7200	1.4030	0.6980	-0.3700	-0.5510

1.4.2.1.2. Graphical Output and Interpretation

Goal

The goal of this analysis is threefold:

1. Determine if the univariate model:

   $$Y_i = C + E_i$$

   is appropriate and valid.

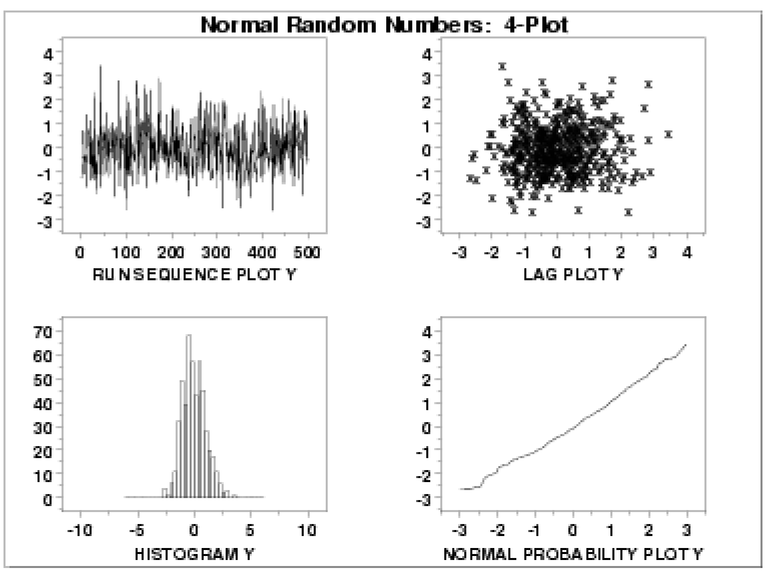
2. Determine if the typical underlying assumptions for an "in control" measurement process are valid. These assumptions are:
   1. random drawings;
   2. from a fixed distribution;
   3. with the distribution having a fixed location; and
   4. the distribution having a fixed scale.

3. Determine if the confidence interval

   $$\bar{Y} \pm 2s/\sqrt{N}$$

   is appropriate and valid where ***s*** is the standard deviation of the original data.

4-Plot of Data



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates that the data do not have any significant shifts in location or scale over time. The run sequence plot does not show any obvious outliers.

2. The lag plot (upper right) does not indicate any non-random pattern in the data.

3. The histogram (lower left) shows that the data are reasonably symmetric, there do not appear to be significant outliers in the tails, and that it is reasonable to assume that the data are from approximately a normal distribution.

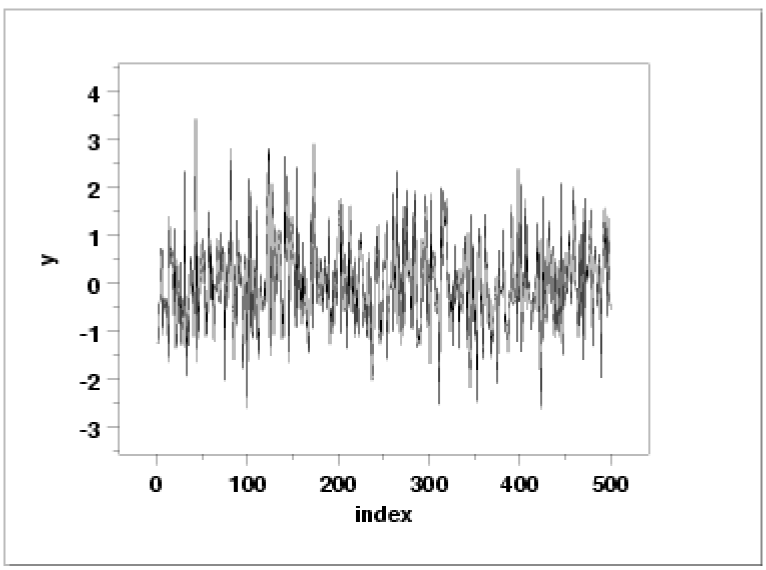
4. The normal probability plot (lower right) verifies that an assumption of normality is in fact reasonable.

From the above plots, we conclude that the underlying assumptions are valid and the data follow approximately a normal distribution. Therefore, the confidence interval form given previously is appropriate for quantifying the uncertainty of the population mean. The numerical values for this model are given in the Quantitative Output and Interpretation section.

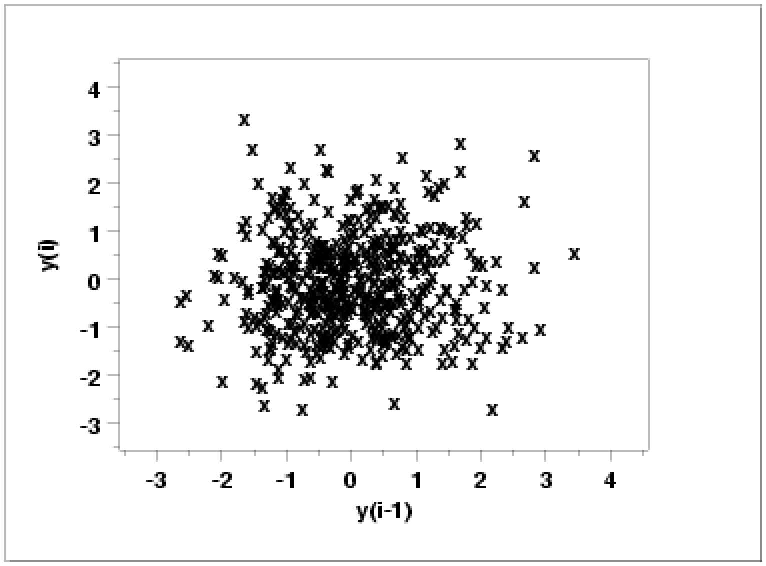
Individual Plots

Although it is usually not necessary, the plots can be generated individually to give more detail.

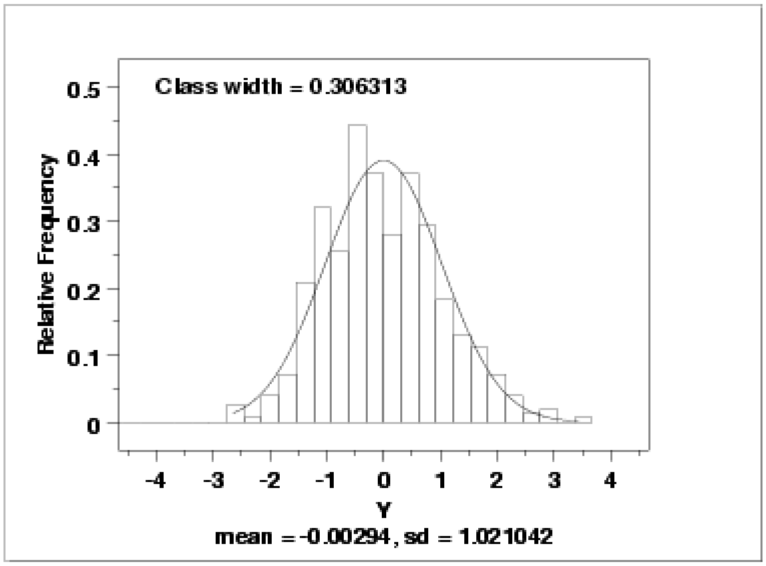
Run Sequence Plot



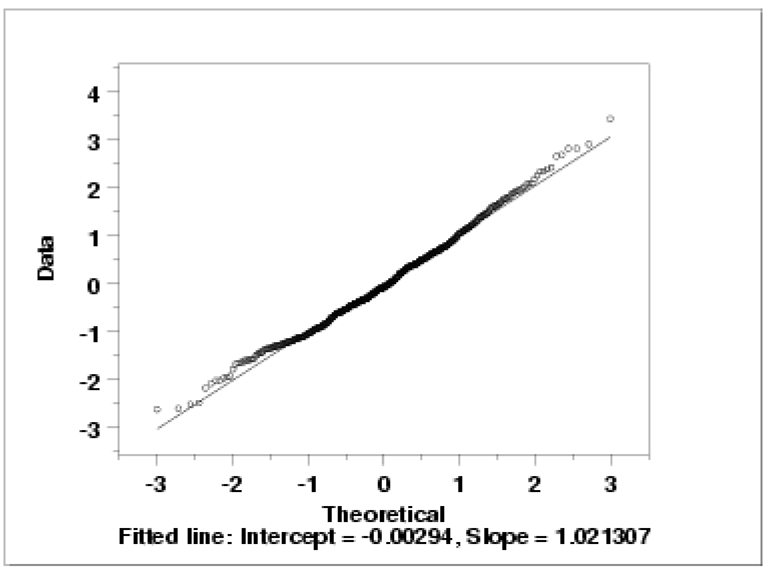
Lag Plot



Histogram (with overlaid Normal PDF)



Normal Probability Plot



1.4.2.1.3. Quantitative Output and Interpretation

Summary Statistics

As a first step in the analysis, common summary statistics are computed from the data.

```
Sample size   = 500
Mean          =  -0.2935997E-02
Median        =  -0.9300000E-01
Minimum       =  -0.2647000E+01
Maximum       =   0.3436000E+01
Range         =   0.6083000E+01
Stan. Dev.    =   0.1021041E+01
```

Location

One way to quantify a change in location over time is to fit a straight line to the data using an index variable as the independent variable in the regression. For our data, we assume that data are in sequential run order and that the data were collected at equally spaced time intervals. In our regression, we use the index variable X = 1, 2, ..., N, where N is the number of observations. If there is no significant drift in the location over time, the slope parameter should be zero.

Coefficient	Estimate	Stan. Error	*t*-Value
B_0	0.699127E-02	0.9155E-01	0.0764
B_1	-0.396298E-04	0.3167E-03	-0.1251

```
Residual Standard Deviation = 1.02205
Residual Degrees of Freedom = 498
```

The absolute value of the *t*-value for the slope parameter is smaller than the critical value of $t_{0.975,498} = 1.96$. Thus, we conclude that the slope is not different from zero at the 0.05 significance level.

Variation

One simple way to detect a change in variation is with Bartlett's test, after dividing the data set into several equal-sized intervals. The choice of the number of intervals is somewhat arbitrary, although values of four or eight are reasonable. We will divide our data into four intervals.

H_0: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$
H_a: At least one σ_i^2 is not equal to the others.

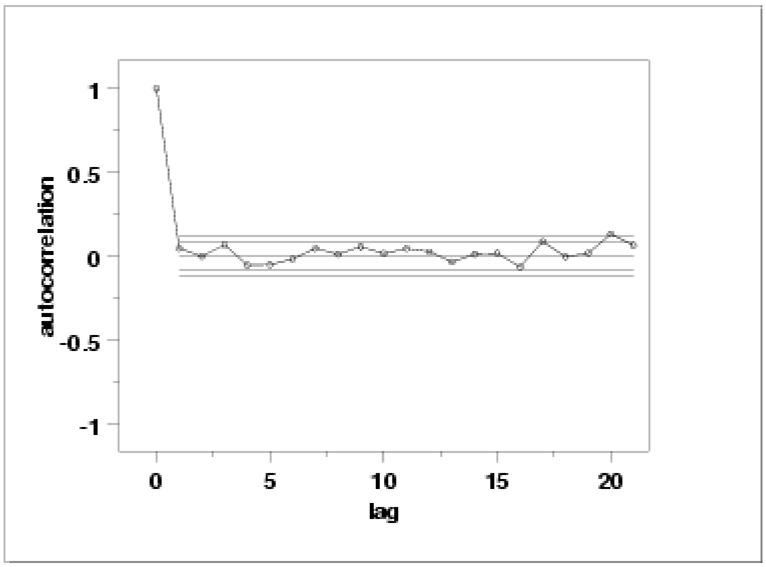
```
Test statistic:   T = 2.373660
Degrees of freedom:   k - 1 = 3
Significance level:   α = 0.05
Critical value:   Χ²_{1-α,k-1} = 7.814728
Critical region:   Reject H0 if T > 7.814728
```

In this case, Bartlett's test indicates that the variances are not significantly different in the four intervals.

Randomness

There are many ways in which data can be non-random. However, most common forms of non-randomness can be detected with a few simple tests including the lag plot shown on the previous page.

Another check is an autocorrelation plot that shows the autocorrelations for various lags. Confidence bands can be plotted at the 95 % and 99 % confidence levels. Points outside this band indicate statistically significant values (lag 0 is always 1).



The lag 1 autocorrelation, which is generally the one of most interest, is 0.045. The critical values at the 5% significance level are -0.087 and 0.087. Since 0.045 is within the critical region, the lag 1 autocorrelation is not statistically significant, so there is no evidence of non-randomness.

A common test for randomness is the runs test.

```
      H0:  the sequence was produced in a random
manner
      Ha:  the sequence was not produced in a
random manner

      Test statistic:   Z = -1.0744
      Significance level:   α = 0.05
      Critical value:   Z_{1-α/2} = 1.96
      Critical region:   Reject H0 if |Z| > 1.96
```

The runs test fails to reject the null hypothesis that the data were produced in a random manner.

Distributional Analysis	Probability plots are a graphical test for assessing if a particular distribution provides an adequate fit to a data set.

A quantitative enhancement to the probability plot is the correlation coefficient of the points on the probability plot, or PPCC. For this data set the PPCC based on a normal distribution is 0.996. Since the PPCC is greater than the critical value of 0.987 (this is a tabulated value), the normality assumption is not rejected.

Chi-square and Kolmogorov-Smirnov goodness-of-fit tests are alternative methods for assessing distributional adequacy. The Wilk-Shapiro and Anderson-Darling tests can be used to test for normality. The results of the Anderson-Darling test follow.

H_0: the data are normally distributed
H_a: the data are not normally distributed

Adjusted test statistic: $A^2 = 1.0612$
Significance level: $\alpha = 0.05$
Critical value: 0.787
Critical region: Reject H_0 if $A^2 > 0.787$

The Anderson-Darling test rejects the normality assumption at the 0.05 significance level.

Outlier Analysis

A test for outliers is the Grubbs test.

H_0: there are no outliers in the data
H_a: the maximum value is an outlier

Test statistic: $G = 3.368068$
Significance level: $\alpha = 0.05$
Critical value for an upper one-tailed test: 3.863087
Critical region: Reject H_0 if $G > 3.863087$

For this data set, Grubbs' test does not detect any outliers at the 0.05 significance level.

Model

Since the underlying assumptions were validated both graphically and analytically, we conclude that a reasonable model for the data is:

$$Y_i = C + E_i$$

where C is the estimated value of the mean, -0.00294. We can express the uncertainty for C as a 95 % confidence interval (-0.09266, 0.08678).

Univariate Report

It is sometimes useful and convenient to summarize the above results in a report.

```
Analysis of 500 normal random numbers

1: Sample Size                          = 500

2: Location
   Mean                                 = -
```

```
0.00294
    Standard Deviation of Mean             =
0.045663
    95% Confidence Interval for Mean       = (-
0.09266,0.086779)
    Drift with respect to location?        = NO

 3: Variation
    Standard Deviation                     =
1.021042
    95% Confidence Interval for SD         =
(0.961437,1.088585)
    Drift with respect to variation?
    (based on Bartletts test on quarters
    of the data)                            = NO

 4: Data are Normal?
      (as tested by Anderson-Darling)      = YES

 5: Randomness
    Autocorrelation                        =
0.045059
    Data are Random?
      (as measured by autocorrelation)     = YES

 6: Statistical Control
    (i.e., no drift in location or scale,
    data are random, distribution is
    fixed, here we are testing only for
    fixed normal)
    Data Set is in Statistical Control?    = YES

 7: Outliers?
    (as determined by Grubbs' test)        = NO
```

1.4.2.1.4. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to repeat the analysis outlined in the case study description on the previous page using Dataplot . It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	Results and Conclusions
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step.	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
1. Invoke Dataplot and read data. 1. Read in the data.	1. You have read 1 column of numbers into Dataplot, variable Y.
2. 4-plot of the data. 1. 4-plot of Y.	1. Based on the 4-plot, there are no shifts in location or scale, and the data seem to follow a normal distribution.
3. Generate the individual plots.	

1. Generate a run sequence plot. 2. Generate a lag plot. 3. Generate a histogram with an overlaid normal pdf. 4. Generate a normal probability plot.	1. The run sequence plot indicates that there are no shifts of location or scale. 2. The lag plot does not indicate any significant patterns (which would show the data were not random). 3. The histogram indicates that a normal distribution is a good distribution for these data. 4. The normal probability plot verifies that the normal distribution is a reasonable distribution for these data.
4. Generate summary statistics, quantitative analysis, and print a univariate report. 1. Generate a table of summary statistics. 2. Generate the mean, a confidence interval for the mean, and compute a linear fit to detect drift in location. 3. Generate the standard deviation, a confidence interval for the standard deviation, and detect drift in variation by dividing the data into quarters and computing Barltett's test for equal standard deviations. 4. Check for randomness by generating an autocorrelation plot and a runs test. 5. Check for normality by computing the normal probability plot correlation coefficient. 6. Check for outliers using Grubbs' test. 7. Print a univariate report (this	1. The summary statistics table displays 25+ statistics. 2. The mean is -0.00294 and a 95% confidence interval is (-0.093,0.087). The linear fit indicates no drift in location since the slope parameter is statistically not significant. 3. The standard deviation is 1.02 with a 95% confidence interval of (0.96,1.09). Bartlett's test indicates no significant change in variation. 4. The lag 1 autocorrelation is 0.04. From the autocorrelation plot, this is within the 95% confidence interval bands.

assumes steps 2 thru 6 have already been run).	5. The normal probability plot correlation coefficient is 0.996. At the 5% level, we cannot reject the normality assumption. 6. Grubbs' test detects no outliers at the 5% level. 7. The results are summarized in a convenient report.

1.4.2.2. Uniform Random Numbers

Uniform Random Numbers

This example illustrates the univariate analysis of a set of uniform random numbers.

1.4.2.2.1. Background and Data

Generation

The uniform random numbers used in this case study are from a Rand Corporation publication.

The motivation for studying a set of uniform random numbers is to illustrate the effects of a known underlying non-normal distribution.

Software

The analyses used in this case study can be generated using both Dataplot code and R code.

Data

The following is the set of uniform random numbers used for this case study.

.100973	.253376	.520135	.863467	.354876
.809590	.911739	.292749	.375420	.480564
.894742	.962480	.524037	.206361	.040200
.822916	.084226	.895319	.645093	.032320
.902560	.159533	.476435	.080336	.990190
.252909	.376707	.153831	.131165	.886767
.439704	.436276	.128079	.997080	.157361
.476403	.236653	.989511	.687712	.171768
.660657	.471734	.072768	.503669	.736170
.658133	.988511	.199291	.310601	.080545
.571824	.063530	.342614	.867990	.743923
.403097	.852697	.760202	.051656	.926866
.574818	.730538	.524718	.623885	.635733
.213505	.325470	.489055	.357548	.284682
.870983	.491256	.737964	.575303	.529647
.783580	.834282	.609352	.034435	.273884
.985201	.776714	.905686	.072210	.940558
.609709	.343350	.500739	.118050	.543139
.808277	.325072	.568248	.294052	.420152
.775678	.834529	.963406	.288980	.831374
.670078	.184754	.061068	.711778	.886854
.020086	.507584	.013676	.667951	.903647
.649329	.609110	.995946	.734887	.517649
.699182	.608928	.937856	.136823	.478341
.654811	.767417	.468509	.505804	.776974
.730395	.718640	.218165	.801243	.563517
.727080	.154531	.822374	.211157	.825314
.385537	.743509	.981777	.402772	.144323
.600210	.455216	.423796	.286026	.699162
.680366	.252291	.483693	.687203	.766211
.399094	.400564	.098932	.050514	.225685
.144642	.756788	.962977	.882254	.382145
.914991	.452368	.479276	.864616	.283554
.947508	.992337	.089200	.803369	.459826
.940368	.587029	.734135	.531403	.334042
.050823	.441048	.194985	.157479	.543297
.926575	.576004	.088122	.222064	.125507
.374211	.100020	.401286	.074697	.966448
.943928	.707258	.636064	.932916	.505344
.844021	.952563	.436517	.708207	.207317
.611969	.044626	.457477	.745192	.433729
.653945	.959342	.582605	.154744	.526695

.270799	.535936	.783848	.823961	.011833
.211594	.945572	.857367	.897543	.875462
.244431	.911904	.259292	.927459	.424811
.621397	.344087	.211686	.848767	.030711
.205925	.701466	.235237	.831773	.208898
.376893	.591416	.262522	.966305	.522825
.044935	.249475	.246338	.244586	.251025
.619627	.933565	.337124	.005499	.765464
.051881	.599611	.963896	.546928	.239123
.287295	.359631	.530726	.898093	.543335
.135462	.779745	.002490	.103393	.598080
.839145	.427268	.428360	.949700	.130212
.489278	.565201	.460588	.523601	.390922
.867728	.144077	.939108	.364770	.617429
.321790	.059787	.379252	.410556	.707007
.867431	.715785	.394118	.692346	.140620
.117452	.041595	.660000	.187439	.242397
.118963	.195654	.143001	.758753	.794041
.921585	.666743	.680684	.962852	.451551
.493819	.476072	.464366	.794543	.590479
.003320	.826695	.948643	.199436	.168108
.513488	.881553	.015403	.545605	.014511
.980862	.482645	.240284	.044499	.908896
.390947	.340735	.441318	.331851	.623241
.941509	.498943	.548581	.886954	.199437
.548730	.809510	.040696	.382707	.742015
.123387	.250162	.529894	.624611	.797524
.914071	.961282	.966986	.102591	.748522
.053900	.387595	.186333	.253798	.145065
.713101	.024674	.054556	.142777	.938919
.740294	.390277	.557322	.709779	.017119
.525275	.802180	.814517	.541784	.561180
.993371	.430533	.512969	.561271	.925536
.040903	.116644	.988352	.079848	.275938
.171539	.099733	.344088	.461233	.483247
.792831	.249647	.100229	.536870	.323075
.754615	.020099	.690749	.413887	.637919
.763558	.404401	.105182	.161501	.848769
.091882	.009732	.825395	.270422	.086304
.833898	.737464	.278580	.900458	.549751
.981506	.549493	.881997	.918707	.615068
.476646	.731895	.020747	.677262	.696229
.064464	.271246	.701841	.361827	.757687
.649020	.971877	.499042	.912272	.953750
.587193	.823431	.540164	.405666	.281310
.030068	.227398	.207145	.329507	.706178
.083586	.991078	.542427	.851366	.158873
.046189	.755331	.223084	.283060	.326481
.333105	.914051	.007893	.326046	.047594
.119018	.538408	.623381	.594136	.285121
.590290	.284666	.879577	.762207	.917575
.374161	.613622	.695026	.390212	.557817
.651483	.483470	.894159	.269400	.397583
.911260	.717646	.489497	.230694	.541374
.775130	.382086	.864299	.016841	.482774
.519081	.398072	.893555	.195023	.717469
.979202	.885521	.029773	.742877	.525165
.344674	.218185	.931393	.278817	.570568

1.4.2.2.2. Graphical Output and Interpretation

Goal

The goal of this analysis is threefold:

1. Determine if the univariate model:

$$Y_i = C + E_i$$

   is appropriate and valid.

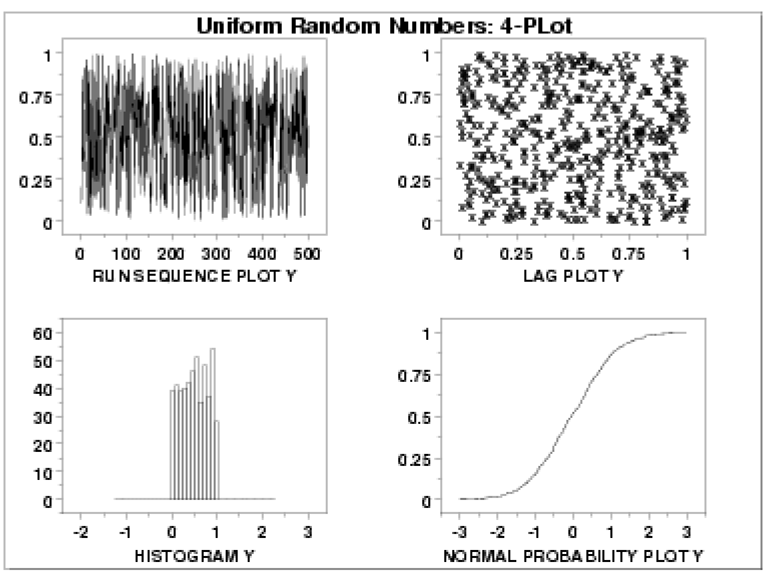
2. Determine if the typical underlying assumptions for an "in control" measurement process are valid. These assumptions are:
   1. random drawings;
   2. from a fixed distribution;
   3. with the distribution having a fixed location; and
   4. the distribution having a fixed scale.

3. Determine if the confidence interval

$$\bar{Y} \pm 2s/\sqrt{N}$$

   is appropriate and valid where ***s*** is the standard deviation of the original data.

4-Plot of Data



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates that the data do not have any significant shifts in location or scale over time.

2. The lag plot (upper right) does not indicate any non-random pattern in the data.

3. The histogram shows that the frequencies are relatively flat across the range of the data. This suggests that the uniform distribution might provide a better distributional fit than the normal distribution.

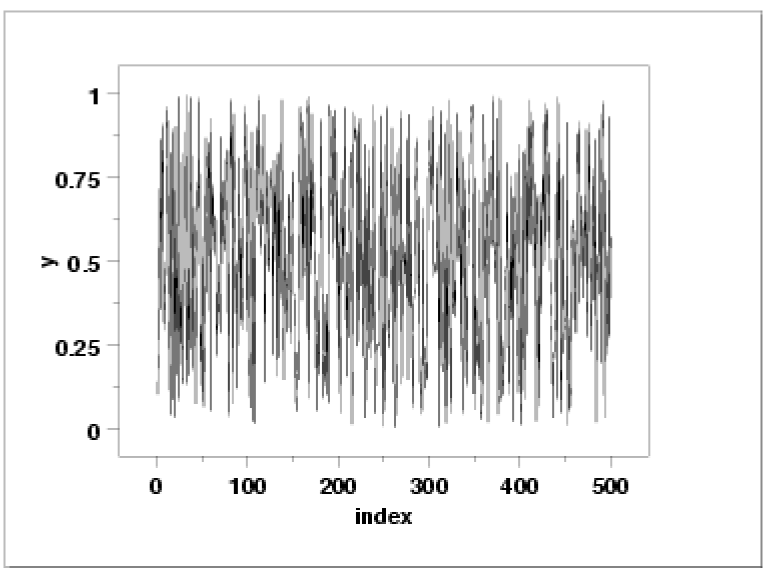
4. The normal probability plot verifies that an assumption of normality is not reasonable. In this case, the 4-plot should be followed up by a uniform probability plot to determine if it provides a better fit to the data. This is shown below.

From the above plots, we conclude that the underlying assumptions are valid. Therefore, the model $Y_i = C + E_i$ is valid. However, since the data are not normally distributed, using the mean as an estimate of C and the confidence interval cited above for quantifying its uncertainty are not valid or appropriate.

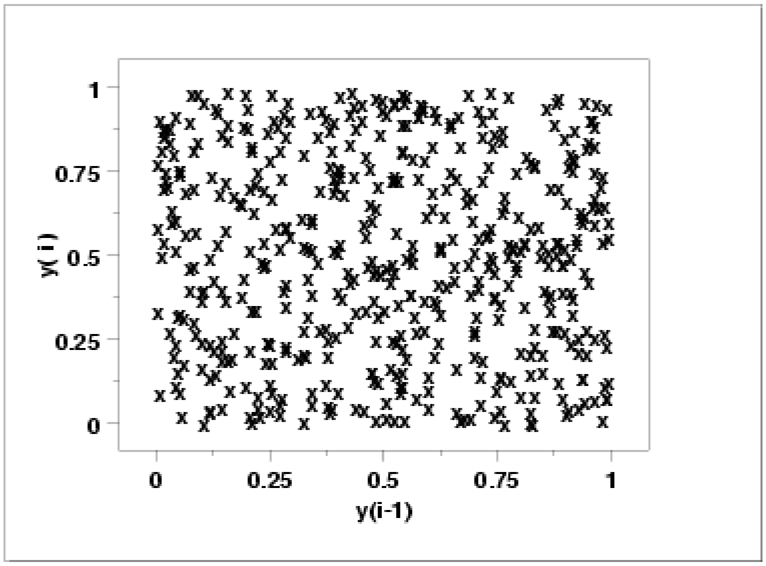
Individual Plots

Although it is usually not necessary, the plots can be generated individually to give more detail.

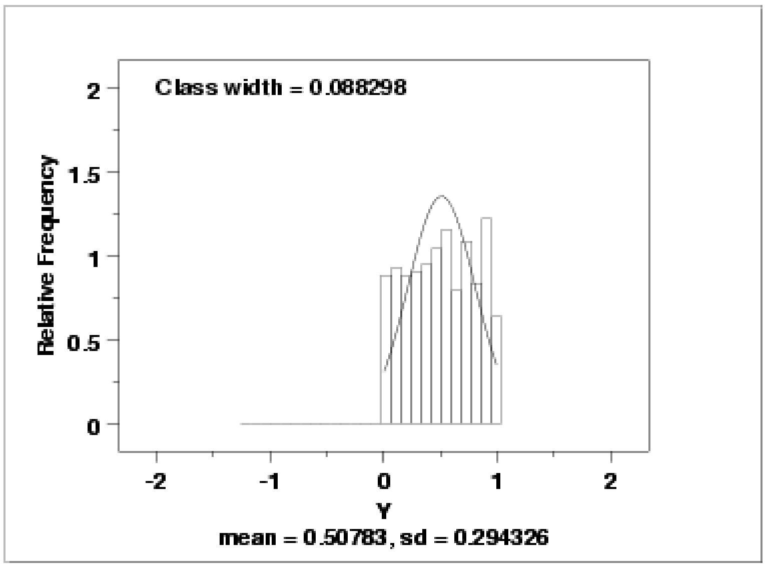
Run Sequence Plot



Lag Plot

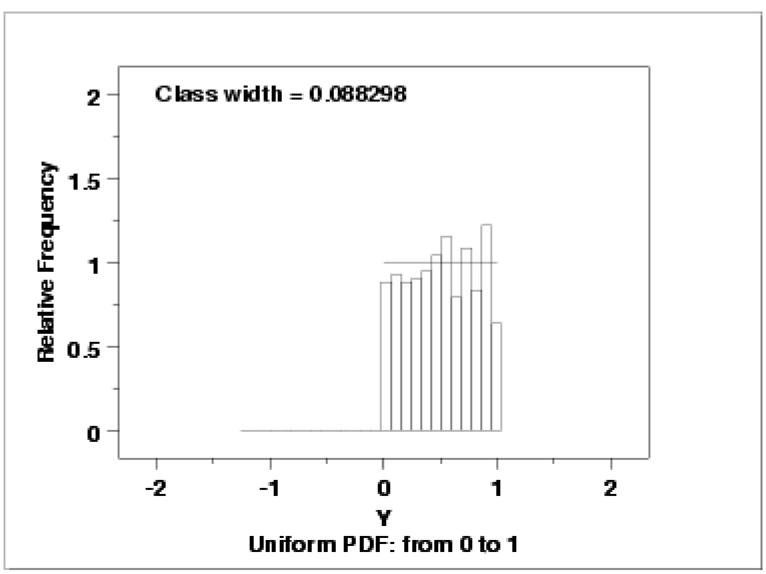


Histogram (with overlaid Normal PDF)



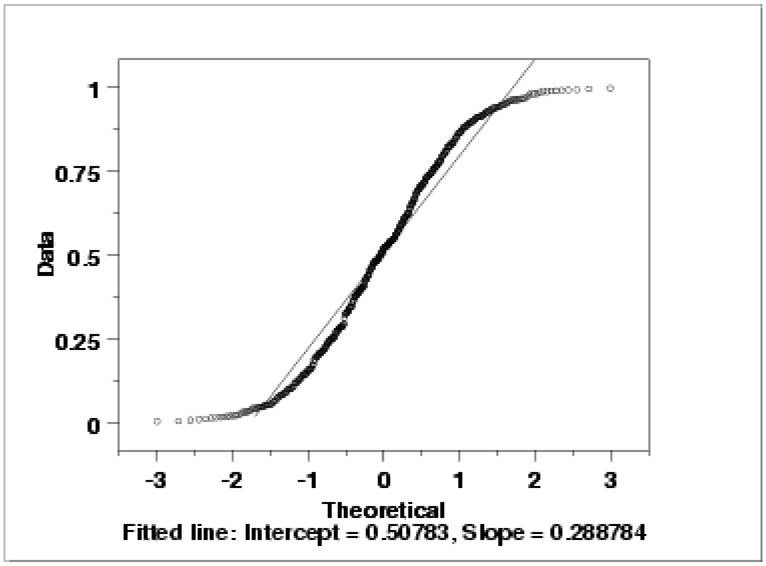
This plot shows that a normal distribution is a poor fit. The flatness of the histogram suggests that a uniform distribution might be a better fit.

Histogram (with overlaid Uniform PDF)



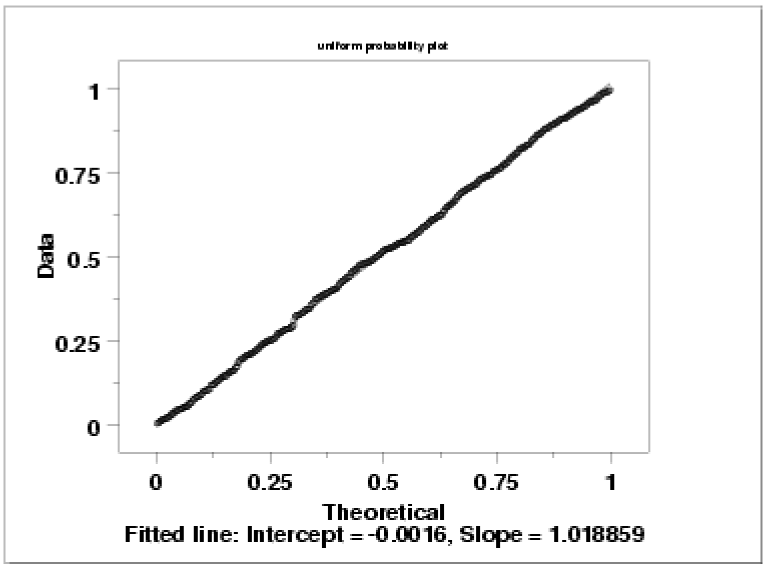
Since the histogram from the 4-plot suggested that the uniform distribution might be a good fit, we overlay a uniform distribution on top of the histogram. This indicates a much better fit than a normal distribution.

Normal Probability Plot



As with the histogram, the normal probability plot shows that the normal distribution does not fit these data well.

Uniform Probability Plot



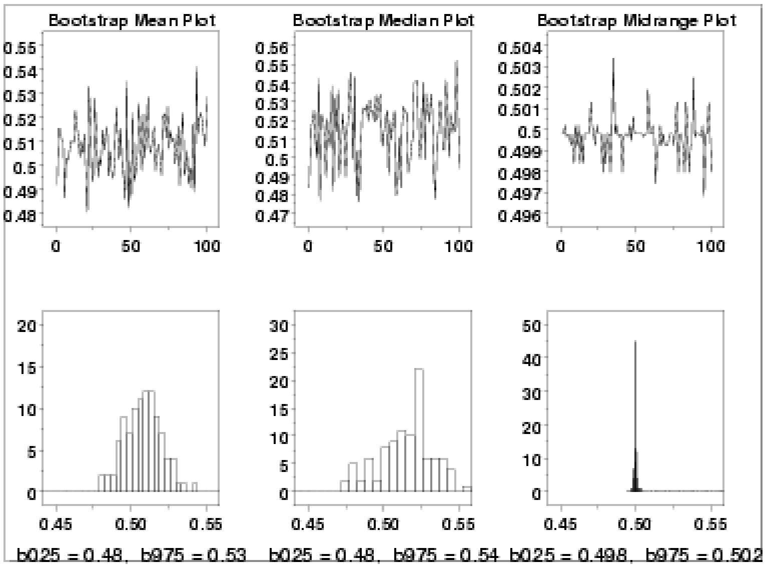
Since the above plots suggested that a uniform distribution might be appropriate, we generate a uniform probability plot. This plot shows that the uniform distribution provides an excellent fit to the data.

Better Model

Since the data follow the underlying assumptions, but with a uniform distribution rather than a normal distribution, we would still like to characterize ***C*** by a typical value plus or minus a confidence interval. In this case, we would like to find a location estimator with the smallest variability.

The bootstrap plot is an ideal tool for this purpose. The following plots show the bootstrap plot, with the corresponding histogram, for the mean, median, mid-range, and median absolute deviation.

Bootstrap Plots



Mid-Range is Best

From the above histograms, it is obvious that for these data, the mid-range is far superior to the mean or median as an estimate for location.

Using the mean, the location estimate is 0.507 and a 95% confidence interval for the mean is (0.482,0.534). Using the mid-range, the location estimate is 0.499 and the 95% confidence interval for the mid-range is (0.497,0.503).

Although the values for the location are similar, the difference in the uncertainty intervals is quite large.

Note that in the case of a uniform distribution it is known theoretically that the mid-range is the best linear unbiased estimator for location. However, in many applications, the most appropriate estimator will not be known or it will be mathematically intractable to determine a valid condfidence interval. The bootstrap provides a method for determining (and comparing) confidence intervals in these cases.

1.4.2.2.3. Quantitative Output and Interpretation

Summary Statistics

As a first step in the analysis, common summary statistics are computed for the data.

```
Sample size   = 500
Mean          =    0.5078304
Median        =    0.5183650
Minimum       =    0.0024900
Maximum       =    0.9970800
Range         =    0.9945900
Stan. Dev.    =    0.2943252
```

Because the graphs of the data indicate the data may not be normally distributed, we also compute two other statistics for the data, the normal PPCC and the uniform PPCC.

```
Normal PPCC  =    0.9771602
Uniform PPCC =    0.9995682
```

The uniform probability plot correlation coefficient (PPCC) value is larger than the normal PPCC value. This is evidence that the uniform distribution fits these data better than does a normal distribution.

Location

One way to quantify a change in location over time is to fit a straight line to the data using an index variable as the independent variable in the regression. For our data, we assume that data are in sequential run order and that the data were collected at equally spaced time intervals. In our regression, we use the index variable X = 1, 2, ..., N, where N is the number of observations. If there is no significant drift in the location over time, the slope parameter should be zero.

Coefficient	Estimate	Stan. Error	t-Value
B_0	0.522923	0.2638E-01	19.82
B_1	-0.602478E-04	0.9125E-04	-0.66

```
Residual Standard Deviation = 0.2944917
Residual Degrees of Freedom = 498
```

The t-value of the slope parameter, -0.66, is smaller than the critical value of $t_{0.975,498}$ = 1.96. Thus, we conclude that the slope is not different from zero at the 0.05

significance level.

Variation

One simple way to detect a change in variation is with a Bartlett test after dividing the data set into several equal-sized intervals. However, the Bartlett test is not robust for non-normality. Since we know this data set is not approximated well by the normal distribution, we use the alternative Levene test. In particular, we use the Levene test based on the median rather the mean. The choice of the number of intervals is somewhat arbitrary, although values of four or eight are reasonable. We will divide our data into four intervals.

H_0: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$
H_a: At least one σ_i^2 is not equal to the others.

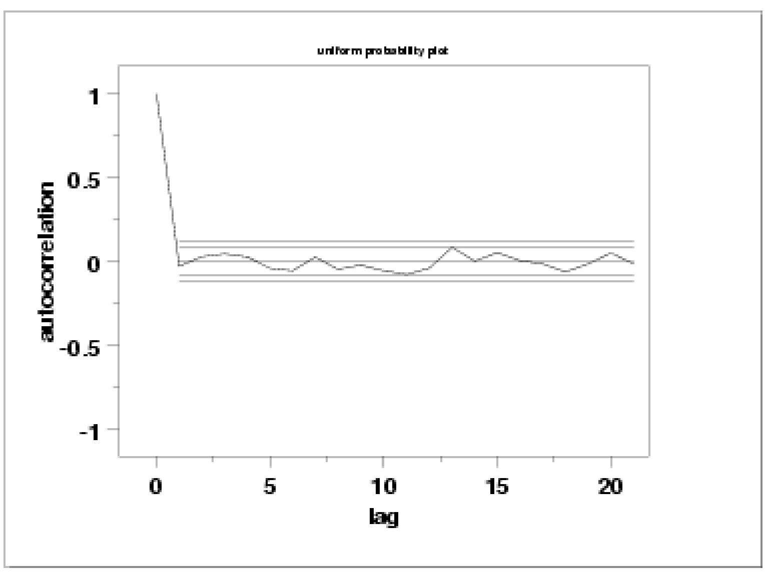
Test statistic: $W = 0.07983$
Degrees of freedom: $k - 1 = 3$
Significance level: $\alpha = 0.05$
Critical value: $F_{\alpha,k-1,N-k} = 2.623$
Critical region: Reject H_0 if $W > 2.623$

In this case, the Levene test indicates that the variances are not significantly different in the four intervals.

Randomness

There are many ways in which data can be non-random. However, most common forms of non-randomness can be detected with a few simple tests including the lag plot shown on the previous page.

Another check is an autocorrelation plot that shows the autocorrelations for various lags. Confidence bands can be plotted using 95% and 99% confidence levels. Points outside this band indicate statistically significant values (lag 0 is always 1).



The lag 1 autocorrelation, which is generally the one of most interest, is 0.03. The critical values at the 5 % significance level are -0.087 and 0.087. This indicates that the lag 1 autocorrelation is not statistically significant, so there is no evidence of non-randomness.

A common test for randomness is the runs test.

```
      H0:  the sequence was produced in a random manner
      Ha:  the sequence was not produced in a random manner

      Test statistic:  Z = 0.2686
      Significance level:  α = 0.05
      Critical value:  Z1-α/2 = 1.96
      Critical region:  Reject H0 if |Z| > 1.96
```

The runs test fails to reject the null hypothesis that the data were produced in a random manner.

Distributional Analysis

Probability plots are a graphical test of assessing whether a particular distribution provides an adequate fit to a data set.

A quantitative enhancement to the probability plot is the correlation coefficient of the points on the probability plot, or PPCC. For this data set the PPCC based on a normal distribution is 0.977. Since the PPCC is less than the critical value of 0.987 (this is a tabulated value), the normality assumption is rejected.

Chi-square and Kolmogorov-Smirnov goodness-of-fit tests are alternative methods for assessing distributional adequacy. The Wilk-Shapiro and Anderson-Darling tests can be used to test for normality. The results of the Anderson-Darling test follow.

```
      H0:  the data are normally distributed
      Ha:  the data are not normally distributed

      Adjusted test statistic:  A^2 = 5.765
      Significance level:  α = 0.05
      Critical value:  0.787
      Critical region:  Reject H0 if A^2 > 0.787
```

The Anderson-Darling test rejects the normality assumption because the value of the test statistic, 5.765, is larger than the critical value of 0.787 at the 0.05 significance level.

Model

Based on the graphical and quantitative analysis, we use the model

$$Y_i = C + E_i$$

where C is estimated by the mid-range and the uncertainty interval for C is based on a bootstrap analysis. Specifically,

C = 0.499
95% confidence limit for C = (0.497,0.503)

Univariate Report

It is sometimes useful and convenient to summarize the above results in a report.

```
 Analysis for 500 uniform random numbers

 1: Sample Size                                = 500

 2: Location
    Mean                                        =
0.50783
    Standard Deviation of Mean                 =
0.013163
    95% Confidence Interval for Mean           =
(0.48197,0.533692)
    Drift with respect to location?            = NO

 3: Variation
    Standard Deviation                         =
0.294326
    95% Confidence Interval for SD             =
(0.277144,0.313796)
    Drift with respect to variation?
    (based on Levene's test on quarters
    of the data)                                = NO

 4: Distribution
    Normal PPCC                                =
0.9771602
    Data are Normal?
      (as measured by Normal PPCC)             = NO

    Uniform PPCC                               =
0.9995682
    Data are Uniform?
      (as measured by Uniform PPCC)            = YES

 5: Randomness
    Autocorrelation                            = -
0.03099
    Data are Random?
      (as measured by autocorrelation)        = YES

 6: Statistical Control
    (i.e., no drift in location or scale,
    data is random, distribution is
    fixed, here we are testing only for
    fixed uniform)
    Data Set is in Statistical Control?       = YES
```

1.4.2.2.4. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to repeat the analysis outlined in the case study description on the previous page using Dataplot . It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	**Results and Conclusions**
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step.	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
`1. Invoke Dataplot and read data.` `1. Read in the data.`	`1. You have read 1 column of numbers into Dataplot, variable Y.`
`2. 4-plot of the data.` `1. 4-plot of Y.`	`1. Based on the 4-plot, there are no shifts in location or scale, and the data do not seem to follow a normal distribution.`
`3. Generate the individual plots.`	

1. Generate a run sequence plot. 2. Generate a lag plot. 3. Generate a histogram with an overlaid normal pdf. 4. Generate a histogram with an overlaid uniform pdf. 5. Generate a normal probability plot. 6. Generate a uniform probability plot.	1. The run sequence plot indicates that there are no shifts of location or scale. 2. The lag plot does not indicate any significant patterns (which would show the data were not random). 3. The histogram indicates that a normal distribution is not a good distribution for these data. 4. The histogram indicates that a uniform distribution is a good distribution for these data. 5. The normal probability plot verifies that the normal distribution is not a reasonable distribution for these data. 6. The uniform probability plot verifies that the uniform distribution is a reasonable distribution for these data.
4. Generate the bootstrap plot. 1. Generate a bootstrap plot.	1. The bootstrap plot clearly shows the superiority of the mid-range over the mean and median as the location estimator of choice for this problem.
5. Generate summary statistics, quantitative analysis, and print a univariate report. 1. Generate a table of summary statistics. 2. Generate the mean, a confidence interval for the mean, and compute a linear fit to detect drift in location. 3. Generate the standard deviation, a confidence interval for the	1. The summary statistics table displays 25+ statistics. 2. The mean is 0.5078 and a 95% confidence interval is (0.482,0.534). The linear fit indicates no drift in location since the slope parameter

standard deviation, and detect drift in variation by dividing the data into quarters and computing Barltetts test for equal standard deviations.	is statistically not significant.
	3. The standard deviation is 0.29 with a 95% confidence interval of (0.277,0.314). Levene's test indicates no significant drift in variation.
4. Check for randomness by generating an autocorrelation plot and a runs test.	4. The lag 1 autocorrelation is -0.03. From the autocorrelation plot, this is within the 95% confidence interval bands.
5. Check for normality by computing the normal probability plot correlation coefficient.	5. The uniform probability plot correlation coefficient is 0.9995. This indicates that the uniform distribution is a good fit.
6. Print a univariate report (this assumes steps 2 thru 6 have already been run).	6. The results are summarized in a convenient report.

1. Exploratory Data Analysis
1.4. EDA Case Studies
1.4.2. Case Studies

1.4.2.3. Random Walk

Random Walk

This example illustrates the univariate analysis of a set of numbers derived from a random walk.

1.4.2.3.1. Background and Data

Generation

A random walk can be generated from a set of uniform random numbers by the formula:

$$R_i = \sum_{j=1}^{i}(U_j - 0.5)$$

where U is a set of uniform random numbers.

The motivation for studying a set of random walk data is to illustrate the effects of a known underlying autocorrelation structure (i.e., non-randomness) in the data.

Software

The analyses used in this case study can be generated using both Dataplot code and R code.

Data

The following is the set of random walk numbers used for this case study.

```
-0.399027
-0.645651
-0.625516
-0.262049
-0.407173
-0.097583
 0.314156
 0.106905
-0.017675
-0.037111
 0.357631
 0.820111
 0.844148
 0.550509
 0.090709
 0.413625
-0.002149
 0.393170
 0.538263
 0.070583
 0.473143
 0.132676
 0.109111
-0.310553
 0.179637
-0.067454
-0.190747
-0.536916
-0.905751
-0.518984
-0.579280
-0.643004
-1.014925
-0.517845
```

```
-0.860484
-0.884081
-1.147428
-0.657917
-0.470205
-0.798437
-0.637780
-0.666046
-1.093278
-1.089609
-0.853439
-0.695306
-0.206795
-0.507504
-0.696903
-1.116358
-1.044534
-1.481004
-1.638390
-1.270400
-1.026477
-1.123380
-0.770683
-0.510481
-0.958825
-0.531959
-0.457141
-0.226603
-0.201885
-0.078000
 0.057733
-0.228762
-0.403292
-0.414237
-0.556689
-0.772007
-0.401024
-0.409768
-0.171804
-0.096501
-0.066854
 0.216726
 0.551008
 0.660360
 0.194795
-0.031321
 0.453880
 0.730594
 1.136280
 0.708490
 1.149048
 1.258757
 1.102107
 1.102846
 0.720896
 0.764035
 1.072312
 0.897384
 0.965632
 0.759684
 0.679836
 0.955514
 1.290043
 1.753449
 1.542429
 1.873803
 2.043881
 1.728635
 1.289703
 1.501481
 1.888335
 1.408421
 1.416005
 0.929681
 1.097632
 1.501279
 1.650608
 1.759718
 2.255664
 2.490551
 2.508200
```

```
2.707382
2.816310
3.254166
2.890989
2.869330
3.024141
3.291558
3.260067
3.265871
3.542845
3.773240
3.991880
3.710045
4.011288
4.074805
4.301885
3.956416
4.278790
3.989947
4.315261
4.200798
4.444307
4.926084
4.828856
4.473179
4.573389
4.528605
4.452401
4.238427
4.437589
4.617955
4.370246
4.353939
4.541142
4.807353
4.706447
4.607011
4.205943
3.756457
3.482142
3.126784
3.383572
3.846550
4.228803
4.110948
4.525939
4.478307
4.457582
4.822199
4.605752
5.053262
5.545598
5.134798
5.438168
5.397993
5.838361
5.925389
6.159525
6.190928
6.024970
5.575793
5.516840
5.211826
4.869306
4.912601
5.339177
5.415182
5.003303
4.725367
4.350873
4.225085
3.825104
3.726391
3.301088
3.767535
4.211463
4.418722
4.554786
4.987701
4.993045
5.337067
```

```
5.789629
5.726147
5.934353
5.641670
5.753639
5.298265
5.255743
5.500935
5.434664
5.588610
6.047952
6.130557
5.785299
5.811995
5.582793
5.618730
5.902576
6.226537
5.738371
5.449965
5.895537
6.252904
6.650447
7.025909
6.770340
7.182244
6.941536
7.368996
7.293807
7.415205
7.259291
6.970976
7.319743
6.850454
6.556378
6.757845
6.493083
6.824855
6.533753
6.410646
6.502063
6.264585
6.730889
6.753715
6.298649
6.048126
5.794463
5.539049
5.290072
5.409699
5.843266
5.680389
5.185889
5.451353
5.003233
5.102844
5.566741
5.613668
5.352791
5.140087
4.999718
5.030444
5.428537
5.471872
5.107334
5.387078
4.889569
4.492962
4.591042
4.930187
4.857455
4.785815
5.235515
4.865727
4.855005
4.920206
4.880794
4.904395
4.795317
5.163044
4.807122
```

```
5.246230
5.111000
5.228429
5.050220
4.610006
4.489258
4.399814
4.606821
4.974252
5.190037
5.084155
5.276501
4.917121
4.534573
4.076168
4.236168
3.923607
3.666004
3.284967
2.980621
2.623622
2.882375
3.176416
3.598001
3.764744
3.945428
4.408280
4.359831
4.353650
4.329722
4.294088
4.588631
4.679111
4.182430
4.509125
4.957768
4.657204
4.325313
4.338800
4.720353
4.235756
4.281361
3.795872
4.276734
4.259379
3.999663
3.544163
3.953058
3.844006
3.684740
3.626058
3.457909
3.581150
4.022659
4.021602
4.070183
4.457137
4.156574
4.205304
4.514814
4.055510
3.938217
4.180232
3.803619
3.553781
3.583675
3.708286
4.005810
4.419880
4.881163
5.348149
4.950740
5.199262
4.753162
4.640757
4.327090
4.080888
3.725953
3.939054
3.463728
3.018284
```

```
2.661061
3.099980
3.340274
3.230551
3.287873
3.497652
3.014771
3.040046
3.342226
3.656743
3.698527
3.759707
4.253078
4.183611
4.196580
4.257851
4.683387
4.224290
3.840934
4.329286
3.909134
3.685072
3.356611
2.956344
2.800432
2.761665
2.744913
3.037743
2.787390
2.387619
2.424489
2.247564
2.502179
2.022278
2.213027
2.126914
2.264833
2.528391
2.432792
2.037974
1.699475
2.048244
1.640126
1.149858
1.475253
1.245675
0.831979
1.165877
1.403341
1.181921
1.582379
1.632130
2.113636
2.163129
2.545126
2.963833
3.078901
3.055547
3.287442
2.808189
2.985451
3.181679
2.746144
2.517390
2.719231
2.581058
2.838745
2.987765
3.459642
3.458684
3.870956
4.324706
4.411899
4.735330
4.775494
4.681160
4.462470
3.992538
3.719936
3.427081
3.256588
```

```
3.462766
3.046353
3.537430
3.579857
3.931223
3.590096
3.136285
3.391616
3.114700
2.897760
2.724241
2.557346
2.971397
2.479290
2.305336
1.852930
1.471948
1.510356
1.633737
1.727873
1.512994
1.603284
1.387950
1.767527
2.029734
2.447309
2.321470
2.435092
2.630118
2.520330
2.578147
2.729630
2.713100
3.107260
2.876659
2.774242
3.185503
3.403148
3.392646
3.123339
3.164713
3.439843
3.321929
3.686229
3.203069
3.185843
3.204924
3.102996
3.496552
3.191575
3.409044
3.888246
4.273767
3.803540
4.046417
4.071581
3.916256
3.634441
4.065834
3.844651
3.915219
```

1.4.2.3.2. Test Underlying Assumptions

Goal

The goal of this analysis is threefold:

1. Determine if the univariate model:

   $$Y_i = C + E_i$$

   is appropriate and valid.

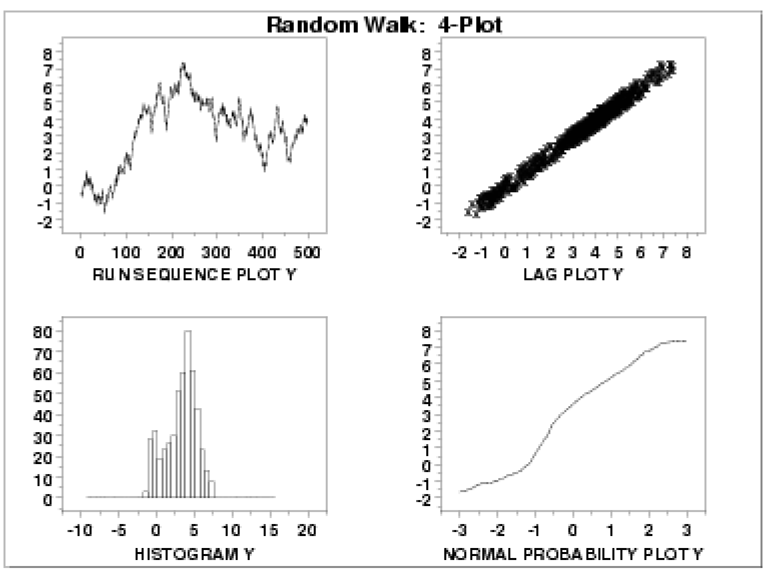
2. Determine if the typical underlying assumptions for an "in control" measurement process are valid. These assumptions are:
   1. random drawings;
   2. from a fixed distribution;
   3. with the distribution having a fixed location; and
   4. the distribution having a fixed scale.

3. Determine if the confidence interval

   $$\bar{Y} \pm 2s/\sqrt{N}$$

   is appropriate and valid, with ***s*** denoting the standard deviation of the original data.

4-Plot of Data



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates significant shifts in location over time.

2. The lag plot (upper right) indicates significant non-randomness in the data.

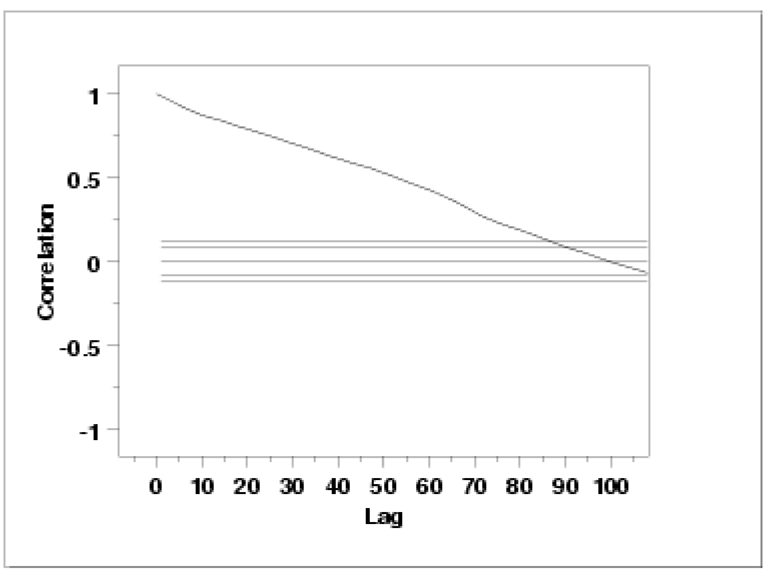
3. When the assumptions of randomness and constant location and scale are not satisfied, the distributional assumptions are not meaningful. Therefore we do not attempt to make any interpretation of the histogram (lower left) or the normal probability plot (lower right).

From the above plots, we conclude that the underlying assumptions are seriously violated. Therefore the $Y_i = C + E_i$ model is not valid.

When the randomness assumption is seriously violated, a time series model may be appropriate. The lag plot often suggests a reasonable model. For example, in this case the strongly linear appearance of the lag plot suggests a model fitting Y_i versus Y_{i-1} might be appropriate. When the data are non-random, it is helpful to supplement the lag plot with an autocorrelation plot and a spectral plot. Although in this case the lag plot is enough to suggest an appropriate model, we provide the autocorrelation and spectral plots for comparison.

Autocorrelation Plot

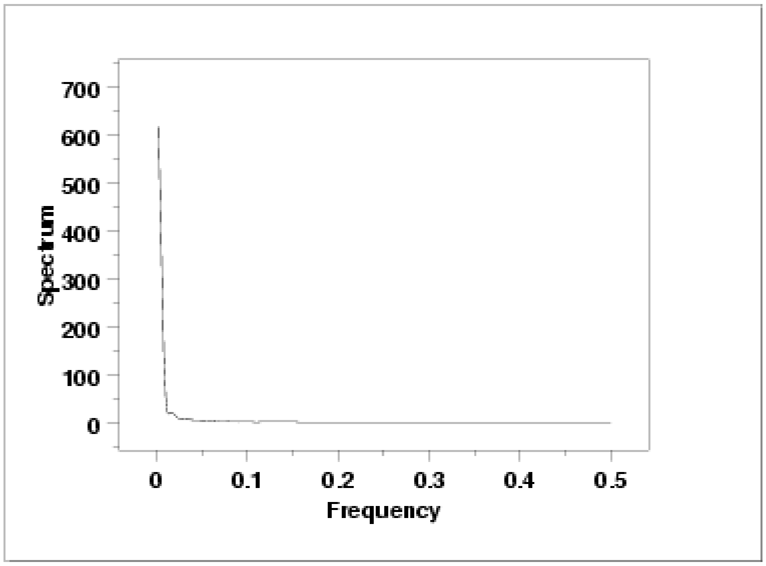
When the lag plot indicates significant non-randomness, it can be helpful to follow up with a an autocorrelation plot.



This autocorrelation plot shows significant autocorrelation at lags 1 through 100 in a linearly decreasing fashion.

Spectral Plot

Another useful plot for non-random data is the spectral plot.



This spectral plot shows a single dominant low frequency peak.

Quantitative Output

Although the 4-plot above clearly shows the violation of the assumptions, we supplement the graphical output with some quantitative measures.

Summary Statistics

As a first step in the analysis, common summary statistics are computed from the data.

```
Sample size  = 500
Mean         =   3.216681
Median       =   3.612030
Minimum      =  -1.638390
Maximum      =   7.415205
Range        =   9.053595
Stan. Dev.   =   2.078675
```

We also computed the autocorrelation to be 0.987, which is evidence of a very strong autocorrelation.

Location

One way to quantify a change in location over time is to fit a straight line to the data using an index variable as the independent variable in the regression. For our data, we assume that data are in sequential run order and that the data were collected at equally spaced time intervals. In our regression, we use the index variable X = 1, 2, ..., N, where N is the number of observations. If there is no significant drift in the location over time, the slope parameter should be zero.

Coefficient	Estimate	Stan. Error	*t*-Value
B_0	1.83351	0.1721	10.650
B_1	0.552164E-02	0.5953E-03	9.275

```
Residual Standard Deviation = 1.9214
Residual Degrees of Freedom = 498
```

The *t*-value of the slope parameter, 9.275, is larger than the critical value of $t_{0.975,498}$ = 1.96. Thus, we conclude that the slope is different from zero at the 0.05 significance level.

Variation

One simple way to detect a change in variation is with a Bartlett test after dividing the data set into several equal-sized intervals. However, the Bartlett test is not robust for non-normality. Since we know this data set is not approximated well by the normal distribution, we use the alternative Levene test. In particular, we use the Levene test based on the median rather the mean. The choice of the number of intervals is somewhat arbitrary, although values of four or eight are reasonable. We will divide our data into four intervals.

H_0: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$
H_a: At least one σ_i^2 is not equal to the others.

Test statistic: W = 10.459
Degrees of freedom: $k - 1 = 3$
Significance level: $\alpha = 0.05$
Critical value: $F_{\alpha,k-1,N-k}$ = 2.623
Critical region: Reject H_0 if $W > 2.623$

In this case, the Levene test indicates that the variances are significantly different in the four intervals since the test statistic of 10.459 is greater than the 95 % critical value of 2.623. Therefore we conclude that the scale is not constant.

Randomness

Although the lag 1 autocorrelation coefficient above clearly shows the non-randomness, we show the output from a runs test as well.

H_0: the sequence was produced in a random manner
H_a: the sequence was not produced in a random manner

Test statistic: Z = -20.3239
Significance level: $\alpha = 0.05$
Critical value: $Z_{1-\alpha/2}$ = 1.96
Critical region: Reject H_0 if $|Z| > 1.96$

The runs test rejects the null hypothesis that the data were produced in a random manner at the 0.05 significance level.

Distributional Assumptions

Since the quantitative tests show that the assumptions of randomness and constant location and scale are not met, the distributional measures will not be meaningful. Therefore these quantitative tests are omitted.

1.4.2.3.3. Develop A Better Model

Lag Plot Suggests Better Model

Since the underlying assumptions did not hold, we need to develop a better model.

The lag plot showed a distinct linear pattern. Given the definition of the lag plot, Y_i versus Y_{i-1}, a good candidate model is a model of the form

$$Y_i = A_0 + A_1 * Y_{i-1} + E_i$$

Fit Output

The results of a linear fit of this model generated the following results.

Coefficient	Estimate	Stan. Error	*t*-Value
A_0	0.050165	0.024171	2.075
A_1	0.987087	0.006313	156.350

```
Residual Standard Deviation = 0.2931
Residual Degrees of Freedom = 497
```

The slope parameter, A_1, has a t value of 156.350 which is statistically significant. Also, the residual standard deviation is 0.2931. This can be compared to the standard deviation shown in the summary table, which is 2.078675. That is, the fit to the autoregressive model has reduced the variability by a factor of 7.

Time Series Model

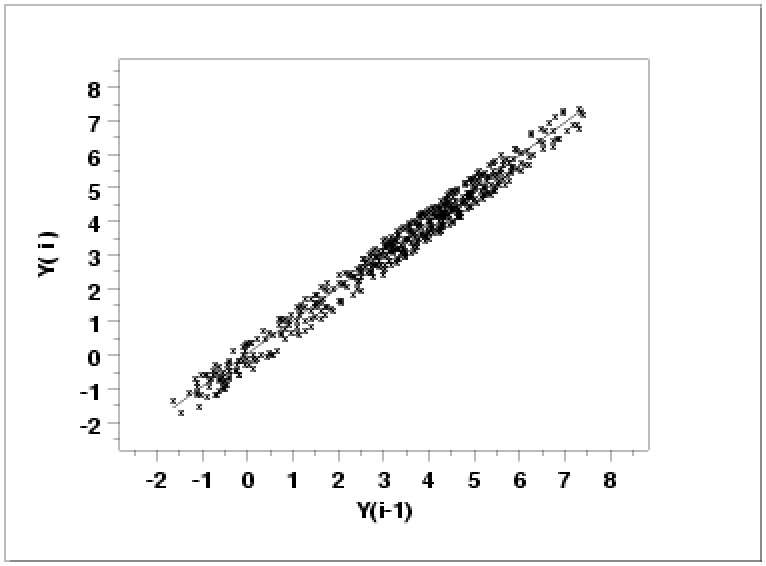
This model is an example of a time series model. More extensive discussion of time series is given in the Process Monitoring chapter.

1. Exploratory Data Analysis
1.4. EDA Case Studies
1.4.2. Case Studies
1.4.2.3. Random Walk

1.4.2.3.4. Validate New Model

Plot Predicted with Original Data

The first step in verifying the model is to plot the predicted values from the fit with the original data.

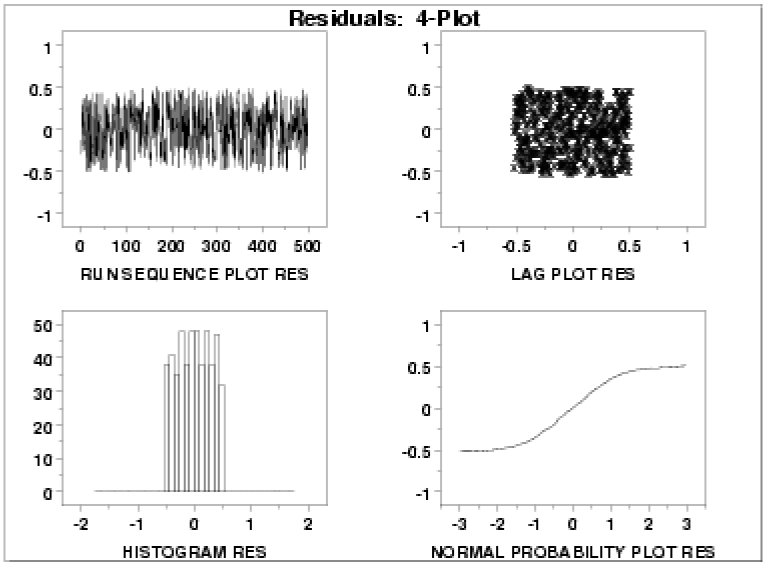


This plot indicates a reasonably good fit.

Test Underlying Assumptions on the Residuals

In addition to the plot of the predicted values, the residual standard deviation from the fit also indicates a significant improvement for the model. The next step is to validate the underlying assumptions for the error component, or residuals, from this model.

4-Plot of Residuals



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates no significant shifts in location or scale over time.

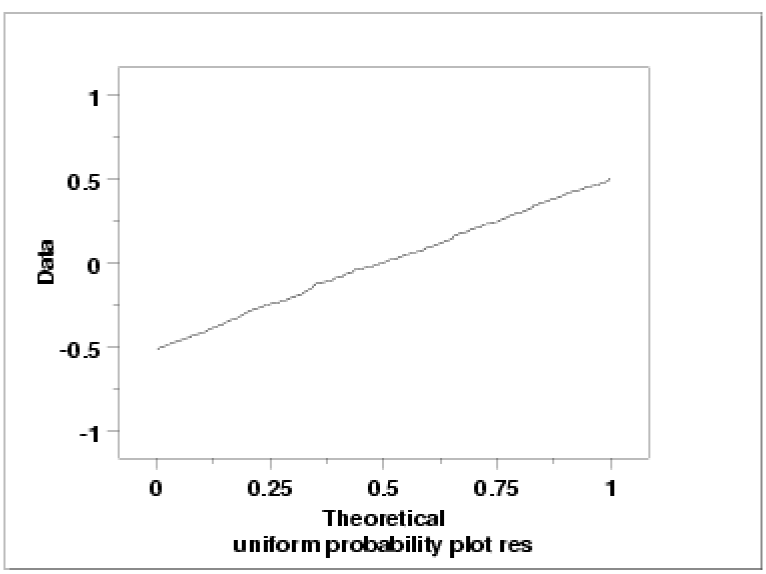
2. The lag plot (upper right) exhibits a random appearance.

3. The histogram shows a relatively flat appearance. This indicates that a uniform probability distribution may be an appropriate model for the error component (or residuals).

4. The normal probability plot clearly shows that the normal distribution is not an appropriate model for the error component.

A uniform probability plot can be used to further test the suggestion that a uniform distribution might be a good model for the error component.

Uniform Probability Plot of Residuals



Since the uniform probability plot is nearly linear, this verifies that a uniform distribution is a good model for the error component.

Conclusions

Since the residuals from our model satisfy the underlying assumptions, we conlude that

$$Y_i = 0.0502 + 0.987 * Y_{i-1} + E_i$$

where the E_i follow a uniform distribution is a good model for this data set. We could simplify this model to

$$Y_i = 1.0 * Y_{i-1} + E_i$$

This has the advantage of simplicity (the current point is simply the previous point plus a uniformly distributed error term).

Using Scientific and Engineering Knowledge

In this case, the above model makes sense based on our definition of the random walk. That is, a random walk is the cumulative sum of uniformly distributed data points. It makes sense that modeling the current point as the previous point plus a uniformly distributed error term is about as good as we can do. Although this case is a bit artificial in that we knew how the data were constructed, it is common and desirable to use scientific and engineering knowledge of the process that generated the data in formulating and testing models for the data. Quite often, several competing models will produce nearly equivalent mathematical results. In this case, selecting the model that best approximates the scientific understanding of the process is a reasonable choice.

Time Series Model

This model is an example of a time series model. More extensive discussion of time series is given in the Process Monitoring chapter.

1.4.2.3.5. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to repeat the analysis outlined in the case study description on the previous page using Dataplot . It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	Results and Conclusions
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step.	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
1. Invoke Dataplot and read data. 1. Read in the data.	1. You have read 1 column of numbers into Dataplot, variable Y.
2. Validate assumptions. 1. 4-plot of Y. 2. Generate a table of summary statistics. 3. Generate a linear fit to detect drift in location. 4. Detect drift in variation by	1. Based on the 4-plot, there are shifts in location and scale and the data are not random. 2. The summary statistics table displays 25+ statistics.

dividing the data into quarters and computing Levene's test for equal standard deviations. 5. Check for randomness by generating a runs test.	3. The linear fit indicates drift in location since the slope parameter is statistically significant. 4. Levene's test indicates significant drift in variation. 5. The runs test indicates significant non-randomness.
3. Generate the randomness plots. 1. Generate an autocorrelation plot. 2. Generate a spectral plot.	1. The autocorrelation plot shows significant autocorrelation at lag 1. 2. The spectral plot shows a single dominant low frequency peak.
4. Fit $Y_i = A0 + A1*Y_{i-1} + E_i$ and validate. 1. Generate the fit. 2. Plot fitted line with original data. 3. Generate a 4-plot of the residuals from the fit. 4. Generate a uniform probability plot of the residuals.	1. The residual standard deviation from the fit is 0.29 (compared to the standard deviation of 2.08 from the original data). 2. The plot of the predicted values with the original data indicates a good fit. 3. The 4-plot indicates that the assumptions of constant location and scale are valid. The lag plot indicates that the data are random. However, the histogram and normal probability plot indicate that the uniform disribution might be a better model for the residuals than the normal distribution. 4. The uniform

	probability plot verifies that the residuals can be fit by a uniform distribution.

1.4.2.4. Josephson Junction Cryothermometry

Josephson Junction Cryothermometry

This example illustrates the univariate analysis of Josephson junction cyrothermometry.

1.4.2.4.1. Background and Data

Generation This data set was collected by Bob Soulen of NIST in October, 1971 as a sequence of observations collected equi-spaced in time from a volt meter to ascertain the process temperature in a Josephson junction cryothermometry (low temperature) experiment. The response variable is voltage counts.

Motivation The motivation for studying this data set is to illustrate the case where there is discreteness in the measurements, but the underlying assumptions hold. In this case, the discreteness is due to the data being integers.

Software The analyses used in this case study can be generated using both Dataplot code and R code.

Data The following are the data used for this case study.

```
2899 2898 2898 2900 2898
2901 2899 2901 2900 2898
2898 2898 2898 2900 2898
2897 2899 2897 2899 2899
2900 2897 2900 2900 2899
2898 2898 2899 2899 2899
2899 2899 2898 2899 2899
2899 2902 2899 2900 2898
2899 2899 2899 2899 2899
2899 2900 2899 2900 2898
2901 2900 2899 2899 2899
2899 2899 2900 2899 2898
2898 2898 2900 2896 2897
2899 2899 2900 2898 2900
2901 2898 2899 2901 2900
2898 2900 2899 2899 2897
2899 2898 2899 2899 2898
2899 2897 2899 2899 2897
2899 2897 2899 2897 2897
2899 2897 2898 2898 2899
2897 2898 2897 2899 2899
2898 2898 2897 2898 2895
2897 2898 2898 2896 2898
2898 2897 2896 2898 2898
2897 2897 2898 2898 2896
2898 2898 2896 2899 2898
2898 2898 2899 2899 2898
2898 2899 2899 2899 2900
2900 2901 2899 2898 2898
2900 2899 2898 2901 2897
2898 2898 2900 2899 2899
2898 2898 2899 2898 2901
2900 2897 2897 2898 2898
2900 2898 2899 2898 2898
2898 2896 2895 2898 2898
2898 2898 2897 2897 2895
```

2897	2897	2900	2898	2896
2897	2898	2898	2899	2898
2897	2898	2898	2896	2900
2899	2898	2896	2898	2896
2896	2896	2897	2897	2896
2897	2897	2896	2898	2896
2898	2896	2897	2896	2897
2897	2898	2897	2896	2895
2898	2896	2896	2898	2896
2898	2898	2897	2897	2898
2897	2899	2896	2897	2899
2900	2898	2898	2897	2898
2899	2899	2900	2900	2900
2900	2899	2899	2899	2898
2900	2901	2899	2898	2900
2901	2901	2900	2899	2898
2901	2899	2901	2900	2901
2898	2900	2900	2898	2900
2900	2898	2899	2901	2900
2899	2899	2900	2900	2899
2900	2901	2899	2898	2898
2899	2896	2898	2897	2898
2898	2897	2897	2897	2898
2897	2899	2900	2899	2897
2898	2900	2900	2898	2898
2899	2900	2898	2900	2900
2898	2900	2898	2898	2898
2898	2898	2899	2898	2900
2897	2899	2898	2899	2898
2897	2900	2901	2899	2898
2898	2901	2898	2899	2897
2899	2897	2896	2898	2898
2899	2900	2896	2897	2897
2898	2899	2899	2898	2898
2897	2897	2898	2897	2897
2898	2898	2898	2896	2895
2898	2898	2898	2896	2898
2898	2898	2897	2897	2899
2896	2900	2897	2897	2898
2896	2897	2898	2898	2898
2897	2897	2898	2899	2897
2898	2899	2897	2900	2896
2899	2897	2898	2897	2900
2899	2900	2897	2897	2898
2897	2899	2899	2898	2897
2901	2900	2898	2901	2899
2900	2899	2898	2900	2900
2899	2898	2897	2900	2898
2898	2897	2899	2898	2900
2899	2898	2899	2897	2900
2898	2902	2897	2898	2899
2899	2899	2898	2897	2898
2897	2898	2899	2900	2900
2899	2898	2899	2900	2899
2900	2899	2899	2899	2899
2899	2898	2899	2899	2900
2902	2899	2900	2900	2901
2899	2901	2899	2899	2902
2898	2898	2898	2898	2899
2899	2900	2900	2900	2898
2899	2899	2900	2899	2900
2899	2900	2898	2898	2898
2900	2898	2899	2900	2899
2899	2900	2898	2898	2899
2899	2899	2899	2898	2898
2897	2898	2899	2897	2897
2901	2898	2897	2898	2899
2898	2897	2899	2898	2897
2898	2898	2897	2898	2899
2899	2899	2899	2900	2899
2899	2897	2898	2899	2900
2898	2897	2901	2899	2901
2898	2899	2901	2900	2900
2899	2900	2900	2900	2900
2901	2900	2901	2899	2897
2900	2900	2901	2899	2898
2900	2899	2899	2900	2899
2900	2899	2900	2899	2901
2900	2900	2899	2899	2898
2899	2900	2898	2899	2899
2901	2898	2898	2900	2899

```
2899 2898 2897 2898 2897
2899 2899 2899 2898 2898
2897 2898 2899 2897 2897
2899 2898 2898 2899 2899
2901 2899 2899 2899 2897
2900 2896 2898 2898 2900
2897 2899 2897 2896 2898
2897 2898 2899 2896 2899
2901 2898 2898 2896 2897
2899 2897 2898 2899 2898
2898 2898 2898 2898 2898
2899 2900 2899 2901 2898
2899 2899 2898 2900 2898
2899 2899 2901 2900 2901
2899 2901 2899 2901 2899
2900 2902 2899 2898 2899
2900 2899 2900 2900 2901
2900 2899 2901 2901 2899
2898 2901 2897 2898 2901
2900 2902 2899 2900 2898
2900 2899 2900 2899 2899
2899 2898 2900 2898 2899
2899 2899 2899 2898 2900
```

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.4.2.4.2. Graphical Output and Interpretation

Goal

The goal of this analysis is threefold:

1. Determine if the univariate model:

   $$Y_i = C + E_i$$

   is appropriate and valid.

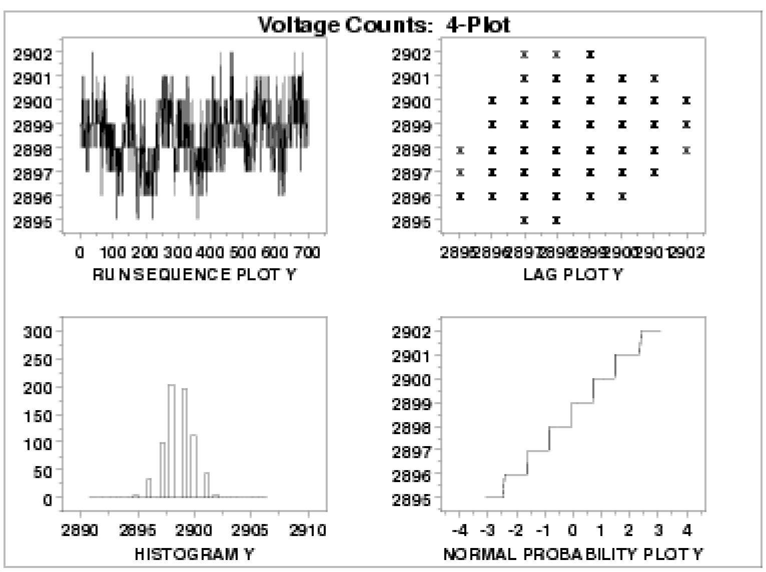
2. Determine if the typical underlying assumptions for an "in control" measurement process are valid. These assumptions are:
   1. random drawings;
   2. from a fixed distribution;
   3. with the distribution having a fixed location; and
   4. the distribution having a fixed scale.

3. Determine if the confidence interval

   $$\bar{Y} \pm 2s/\sqrt{N}$$

   is appropriate and valid where ***s*** is the standard deviation of the original data.

4-Plot of Data



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates that the data do not have any significant shifts in location or scale over time.
2. The lag plot (upper right) does not indicate any non-random pattern in the data.
3. The histogram (lower left) shows that the data are reasonably symmetric, there does not appear to be significant outliers in the tails, and that it is reasonable to assume that the data can be fit with a normal distribution.
4. The normal probability plot (lower right) is difficult to interpret due to the fact that there are only a few distinct values with many repeats.

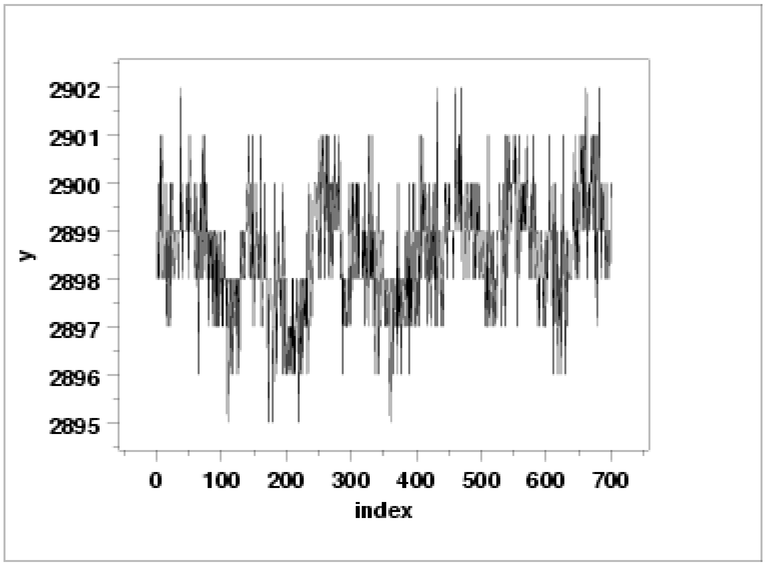
The integer data with only a few distinct values and many repeats accounts for the discrete appearance of several of the plots (e.g., the lag plot and the normal probability plot). In this case, the nature of the data makes the normal probability plot difficult to interpret, especially since each number is repeated many times. However, the histogram indicates that a normal distribution should provide an adequate model for the data.

From the above plots, we conclude that the underlying assumptions are valid and the data can be reasonably approximated with a normal distribution. Therefore, the commonly used uncertainty standard is valid and appropriate. The numerical values for this model are given in the Quantitative Output and Interpretation section.

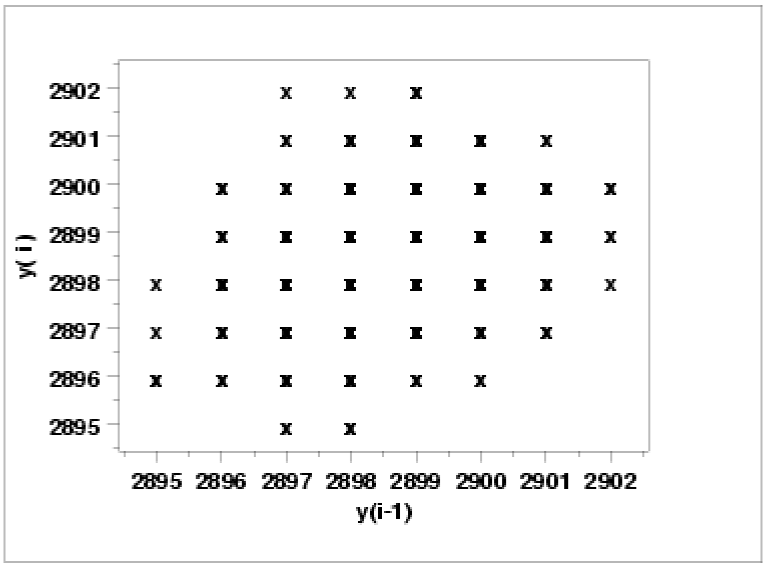
Individual Plots

Although it is normally not necessary, the plots can be generated individually to give more detail.

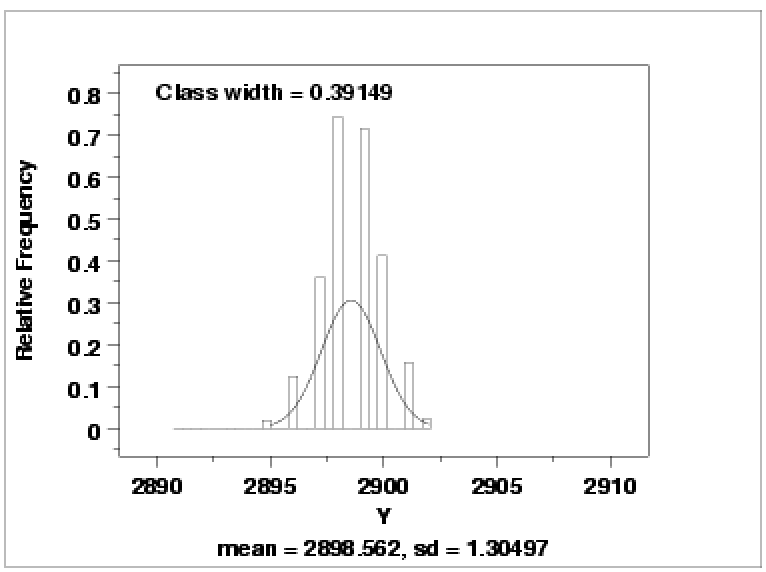
Run Sequence Plot



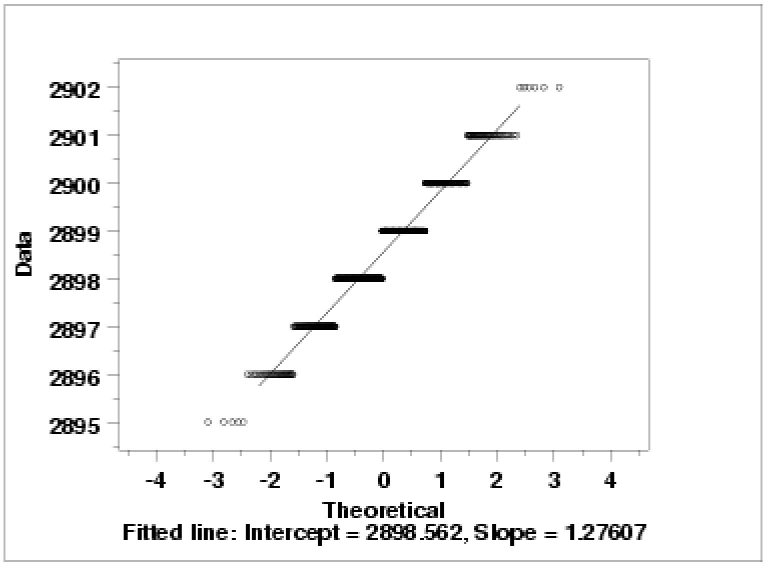
Lag Plot



Histogram (with overlaid Normal PDF)



Normal Probability Plot


2902
2901
2900
2899
2898
2897
2896
2895
Data
-4
-3
-2
-1
0
1
2
3
4
Theoretical
Fitted line: Intercept = 2898.562, Slope = 1.27607

1.4.2.4.3. Quantitative Output and Interpretation

Summary Statistics

As a first step in the analysis, common summary statistics were computed from the data.

```
Sample size  = 700
Mean         =   2898.562
Median       =   2899.000
Minimum      =   2895.000
Maximum      =   2902.000
Range        =      7.000
Stan. Dev.   =      1.305
```

Because of the discrete nature of the data, we also compute the normal PPCC.

```
Normal PPCC = 0.97484
```

Location

One way to quantify a change in location over time is to fit a straight line to the data using an index variable as the independent variable in the regression. For our data, we assume that data are in sequential run order and that the data were collected at equally spaced time intervals. In our regression, we use the index variable X = 1, 2, ..., N, where N is the number of observations. If there is no significant drift in the location over time, the slope parameter should be zero.

Coefficient	Estimate	Stan. Error	*t*-Value
B_0	2.898E+03	9.745E-02	29739.288
B_1	1.071E-03	2.409e-04	4.445

```
Residual Standard Deviation = 1.288
Residual Degrees of Freedom = 698
```

The slope parameter, B_1, has a *t* value of 4.445 which is statistically significant (the critical value is 1.96). However, the value of the slope is 1.071E-03. Given that the slope is nearly zero, the assumption of constant location is not seriously violated even though it is statistically significant.

Variation

One simple way to detect a change in variation is with a Bartlett test after dividing the data set into several equal-sized intervals. However, the Bartlett test is not robust for non-normality. Since the nature of the data (a few distinct points repeated many times) makes the normality assumption questionable, we use the alternative Levene test. In particular, we use the Levene test based on the median rather the mean. The choice of the number of intervals is somewhat arbitrary, although values of four or eight are reasonable. We will divide our data into four intervals.

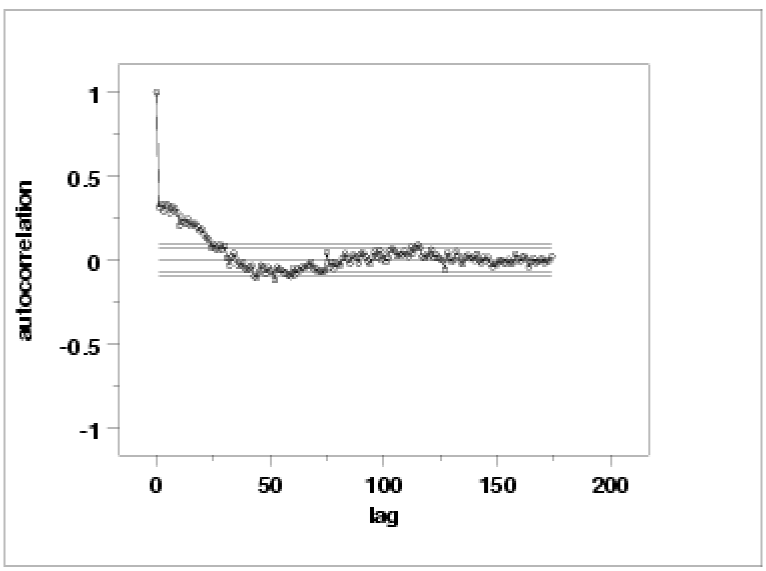
```
      H0:  σ1² = σ2² = σ3² = σ4²
      Ha:  At least one σi² is not equal to the
others.

      Test statistic:  W = 1.43
      Degrees of freedom:  k - 1 = 3
      Significance level:  α = 0.05
      Critical value:  Fα,k-1,N-k = 2.618
      Critical region:  Reject H0 if W > 2.618
```

Since the Levene test statistic value of 1.43 is less than the 95 % critical value of 2.618, we conclude that the variances are not significantly different in the four intervals.

Randomness

There are many ways in which data can be non-random. However, most common forms of non-randomness can be detected with a few simple tests. The lag plot in the previous section is a simple graphical technique.

Another check is an autocorrelation plot that shows the autocorrelations for various lags. Confidence bands can be plotted at the 95 % and 99 % confidence levels. Points outside this band indicate statistically significant values (lag 0 is always 1).



The lag 1 autocorrelation, which is generally the one of most interest, is 0.31. The critical values at the 5 % level of significance are -0.087 and 0.087. This indicates that the lag 1 autocorrelation is statistically significant, so there is some evidence for non-randomness.

A common test for randomness is the runs test.

```
      H0:  the sequence was produced in a random
manner
      Ha:  the sequence was not produced in a
random manner

      Test statistic:  Z = -13.4162
      Significance level:  α = 0.05
      Critical value:  Z1-α/2 = 1.96
      Critical region:  Reject H0 if |Z| > 1.96
```

The runs test indicates non-randomness.

Although the runs test and lag 1 autocorrelation indicate some mild non-randomness, it is not sufficient to reject the $Y_i = C + E_i$ model. At least part of the non-randomness can be explained by the discrete nature of the data.

Distributional Analysis

Probability plots are a graphical test for assessing if a particular distribution provides an adequate fit to a data set.

A quantitative enhancement to the probability plot is the correlation coefficient of the points on the probability plot, or PPCC. For this data set the PPCC based on a normal distribution is 0.975. Since the PPCC is less than the critical value of 0.987 (this is a tabulated value), the normality assumption is rejected.

Chi-square and Kolmogorov-Smirnov goodness-of-fit tests are alternative methods for assessing distributional adequacy. The Wilk-Shapiro and Anderson-Darling tests can be used to test for normality. The results of the Anderson-Darling test follow.

```
      H0:  the data are normally distributed
      Ha:  the data are not normally distributed

      Adjusted test statistic:  A^2 = 16.858
      Significance level:  α = 0.05
      Critical value:  0.787
      Critical region:  Reject H0 if A^2 > 0.787
```

The Anderson-Darling test rejects the normality assumption because the test statistic, 16.858, is greater than the 95 % critical value 0.787.

Although the data are not strictly normal, the violation of the normality assumption is not severe enough to conclude that the $Y_i = C + E_i$ model is unreasonable. At least part of the non-normality can be explained by the discrete nature of the data.

Outlier Analysis

A test for outliers is the Grubbs test.

```
      H0:  there are no outliers in the data
      Ha:  the maximum value is an outlier

      Test statistic:  G = 2.729201
      Significance level:  α = 0.05
      Critical value for a one-tailed test:
```

```
3.950619
      Critical region:  Reject H0 if G > 3.950619
```

For this data set, Grubbs' test does not detect any outliers at the 0.05 significance level.

Model

Although the randomness and normality assumptions were mildly violated, we conclude that a reasonable model for the data is:

$$Y_i = 2898.7 + E_i$$

In addition, a 95 % confidence interval for the mean value is (2898.515, 2898.928).

Univariate Report

It is sometimes useful and convenient to summarize the above results in a report.

```
 Analysis for Josephson Junction Cryothermometry
Data

 1: Sample Size                               = 700

 2: Location
    Mean                                      =
2898.562
    Standard Deviation of Mean                =
0.049323
    95% Confidence Interval for Mean          =
(2898.465,2898.658)
    Drift with respect to location?           = YES
    (Further analysis indicates that
    the drift, while statistically
    significant, is not practically
    significant)

 3: Variation
    Standard Deviation                        =
1.30497
    95% Confidence Interval for SD            =
(1.240007,1.377169)
    Drift with respect to variation?
    (based on Levene's test on quarters
    of the data)                              = NO

 4: Distribution
    Normal PPCC                               =
0.97484
    Data are Normal?
      (as measured by Normal PPCC)            = NO

 5: Randomness
    Autocorrelation                           =
0.314802
    Data are Random?
      (as measured by autocorrelation)        = NO

 6: Statistical Control
    (i.e., no drift in location or scale,
    data are random, distribution is
    fixed, here we are testing only for
    fixed normal)
    Data Set is in Statistical Control?       = NO

    Note: Although we have violations of
    the assumptions, they are mild enough,
    and at least partially explained by the
    discrete nature of the data, so we may model
    the data as if it were in statistical
    control

 7: Outliers?
    (as determined by Grubbs test)            = NO
```

1. Exploratory Data Analysis
1.4. EDA Case Studies
1.4.2. Case Studies
1.4.2.4. Josephson Junction Cryothermometry

1.4.2.4.4. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to repeat the analysis outlined in the case study description on the previous page using Dataplot . It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	**Results and Conclusions**
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step.	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
`1. Invoke Dataplot and read data.` `1. Read in the data.`	`1. You have read 1 column of numbers into Dataplot, variable Y.`
`2. 4-plot of the data.` `1. 4-plot of Y.`	`1. Based on the 4-plot, there are no shifts in location or scale. Due to the nature of the data (a few distinct points with many repeats), the normality assumption is`

	questionable.
3. Generate the individual plots. 1. Generate a run sequence plot. 2. Generate a lag plot. 3. Generate a histogram with an overlaid normal pdf. 4. Generate a normal probability plot.	1. The run sequence plot indicates that there are no shifts of location or scale. 2. The lag plot does not indicate any significant patterns (which would show the data were not random). 3. The histogram indicates that a normal distribution is a good distribution for these data. 4. The discrete nature of the data masks the normality or non-normality of the data somewhat. The plot indicates that a normal distribution provides a rough approximation for the data.
4. Generate summary statistics, quantitative analysis, and print a univariate report. 1. Generate a table of summary statistics. 2. Generate the mean, a confidence interval for the mean, and compute a linear fit to detect drift in location. 3. Generate the standard deviation, a confidence interval for the standard deviation, and detect drift in variation by dividing the data into quarters and computing Levene's test for equal standard deviations. 4. Check for randomness by generating an autocorrelation plot and a runs test. 5. Check for normality by computing the normal probability plot correlation coefficient.	1. The summary statistics table displays 25+ statistics. 2. The mean is 2898.56 and a 95% confidence interval is (2898.46,2898.66). The linear fit indicates no meaningful drift in location since the value of the slope parameter is near zero. 3. The standard devaition is 1.30 with a 95% confidence interval of (1.24,1.38). Levene's test indicates no significant drift in variation.

6. Check for outliers using Grubbs' test.	4. The lag 1 autocorrelation is 0.31. This indicates some mild non-randomness.
7. Print a univariate report (this assumes steps 2 thru 6 have already been run).	5. The normal probability plot correlation coefficient is 0.975. At the 5% level, we reject the normality assumption.
	6. Grubbs' test detects no outliers at the 5% level.
	7. The results are summarized in a convenient report.

1.4.2.5. Beam Deflections

Beam Deflection

This example illustrates the univariate analysis of beam deflection data.

1.4.2.5.1. Background and Data

Generation

This data set was collected by H. S. Lew of NIST in 1969 to measure steel-concrete beam deflections. The response variable is the deflection of a beam from the center point.

The motivation for studying this data set is to show how the underlying assumptions are affected by periodic data.

Data

The following are the data used for this case study.

```
-213
-564
 -35
 -15
 141
 115
-420
-360
 203
-338
-431
 194
-220
-513
 154
-125
-559
  92
 -21
-579
 -52
  99
-543
-175
 162
-457
-346
 204
-300
-474
 164
-107
-572
  -8
  83
-541
-224
 180
-420
-374
 201
-236
-531
  83
  27
-564
-112
 131
```

```
-507
-254
 199
-311
-495
 143
 -46
-579
 -90
 136
-472
-338
 202
-287
-477
 169
-124
-568
  17
  48
-568
-135
 162
-430
-422
 172
 -74
-577
 -13
  92
-534
-243
 194
-355
-465
 156
 -81
-578
 -64
 139
-449
-384
 193
-198
-538
 110
 -44
-577
  -6
  66
-552
-164
 161
-460
-344
 205
-281
-504
 134
 -28
-576
-118
 156
-437
-381
 200
-220
-540
  83
  11
-568
-160
 172
-414
-408
 188
-125
-572
 -32
 139
-492
```

```
-321
 205
-262
-504
 142
 -83
-574
   0
  48
-571
-106
 137
-501
-266
 190
-391
-406
 194
-186
-553
  83
 -13
-577
 -49
 103
-515
-280
 201
 300
-506
 131
 -45
-578
 -80
 138
-462
-361
 201
-211
-554
  32
  74
-533
-235
 187
-372
-442
 182
-147
-566
  25
  68
-535
-244
 194
-351
-463
 174
-125
-570
  15
  72
-550
-190
 172
-424
-385
 198
-218
-536
  96
```

1.4.2.5.2. Test Underlying Assumptions

Goal

The goal of this analysis is threefold:

1. Determine if the univariate model:

   $$Y_i = C + E_i$$

   is appropriate and valid.

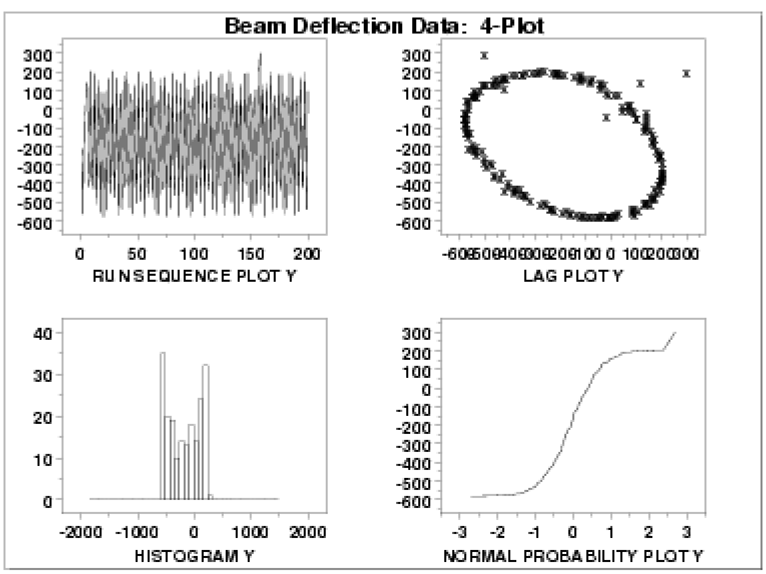
2. Determine if the typical underlying assumptions for an "in control" measurement process are valid. These assumptions are:
   1. random drawings;
   2. from a fixed distribution;
   3. with the distribution having a fixed location; and
   4. the distribution having a fixed scale.

3. Determine if the confidence interval

   $$\bar{Y} \pm 2s/\sqrt{N}$$

   is appropriate and valid where ***s*** is the standard deviation of the original data.

4-Plot of Data



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates that the data do not have any significant shifts in location or scale over time.
2. The lag plot (upper right) shows that the data are not random. The lag plot further indicates the presence of a few outliers.
3. When the randomness assumption is thus seriously violated, the histogram (lower left) and normal probability plot (lower right) are ignored since determining the distribution of the data is only meaningful when the data are random.

From the above plots we conclude that the underlying randomness assumption is not valid. Therefore, the model

$$Y_i = C + E_i$$

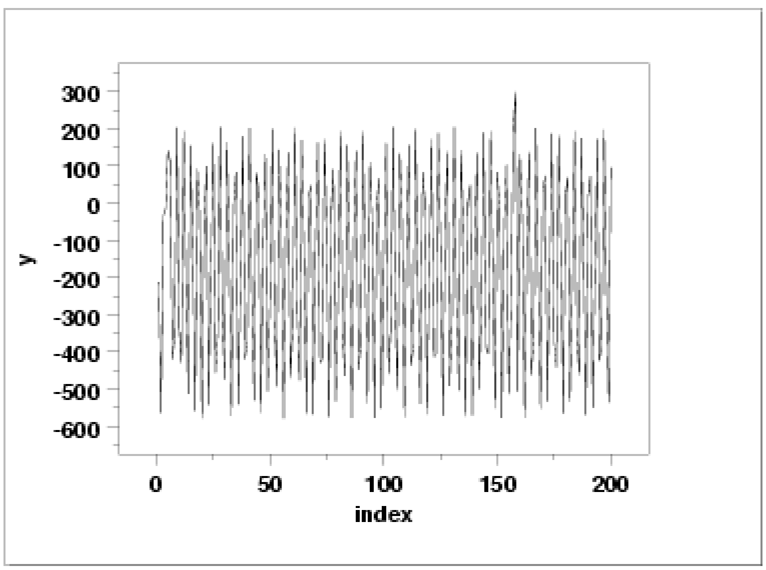
is not appropriate.

We need to develop a better model. Non-random data can frequently be modeled using time series mehtodology. Specifically, the circular pattern in the lag plot indicates that a sinusoidal model might be appropriate. The sinusoidal model will be developed in the next section.

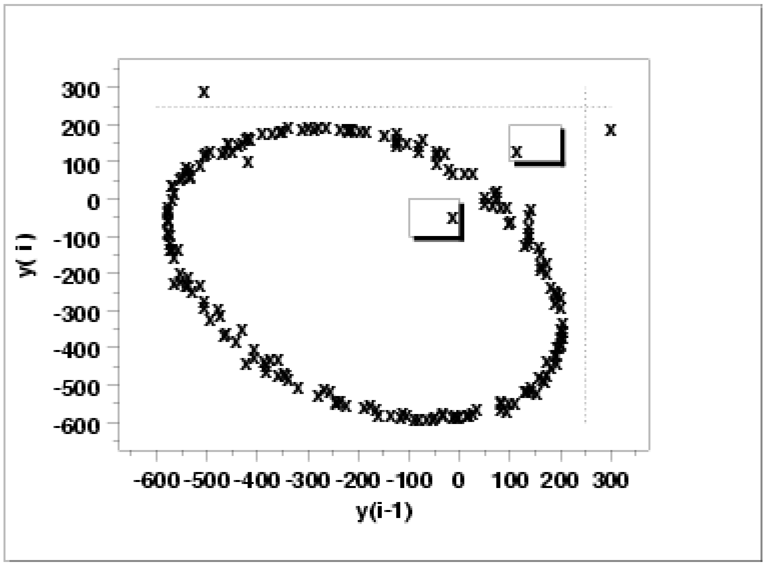
Individual Plots

The plots can be generated individually for more detail. In this case, only the run sequence plot and the lag plot are drawn since the distributional plots are not meaningful.

Run Sequence Plot



Lag Plot



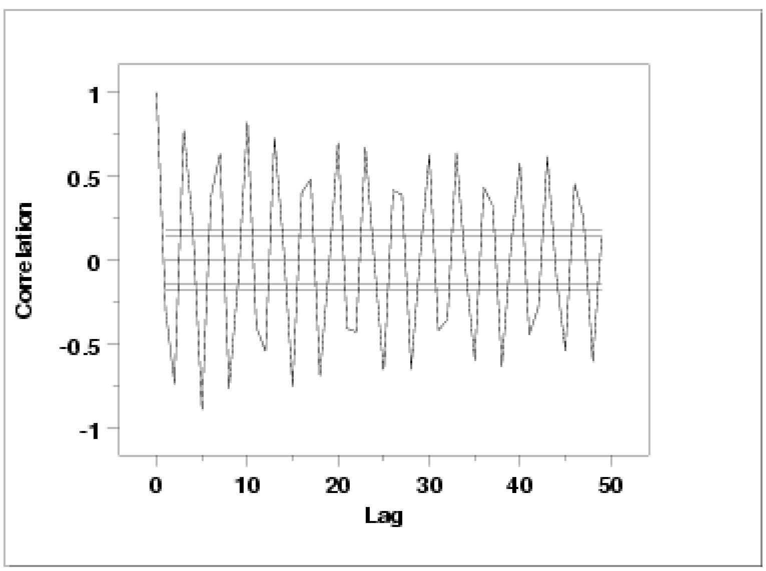
We have drawn some lines and boxes on the plot to better isolate the outliers. The following data points appear to be outliers based on the lag plot.

INDEX	Y(i-1)	Y(i)
158	-506.00	300.00
157	300.00	201.00
3	-15.00	-35.00
5	115.00	141.00

That is, the third, fifth, 157th, and 158th points appear to be outliers.

Autocorrelation Plot

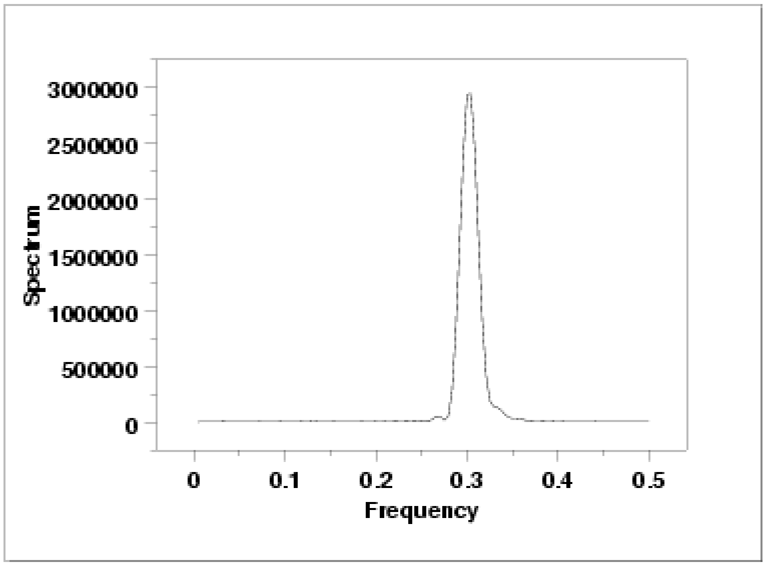
When the lag plot indicates significant non-randomness, it can be helpful to follow up with a an autocorrelation plot.



This autocorrelation plot shows a distinct cyclic pattern. As with the lag plot, this suggests a sinusoidal model.

Spectral Plot

Another useful plot for non-random data is the spectral plot.



This spectral plot shows a single dominant peak at a frequency of 0.3. This frequency of 0.3 will be used in fitting the sinusoidal model in the next section.

Quantitative Results

Although the lag plot, autocorrelation plot, and spectral plot clearly show the violation of the randomness assumption, we supplement the graphical output with some quantitative measures.

Summary Statistics

As a first step in the analysis, summary statistics are computed from the data.

```
Sample size  =  200
Mean         = -177.4350
Median       = -162.0000
Minimum      = -579.0000
Maximum      =  300.0000
Range        =  879.0000
Stan. Dev.   =  277.3322
```

Location

One way to quantify a change in location over time is to fit a straight line to the data set using the index variable X = 1, 2, ..., N, with N denoting the number of observations. If there is no significant drift in the location, the slope parameter should be zero.

Coefficient	Estimate	Stan. Error	*t*-Value
A_0	-178.175	39.47	-4.514
A_1	0.7366E-02	0.34	0.022

```
Residual Standard Deviation = 278.0313
Residual Degrees of Freedom = 198
```

The slope parameter, A1, has a *t* value of 0.022 which is statistically not significant. This indicates that the slope can in fact be considered zero.

Variation

One simple way to detect a change in variation is with a Bartlett test after dividing the data set into several equal-sized intervals. However, the Bartlett the non-randomness of this data does not allows us to assume normality, we use the alternative Levene test. In partiuclar, we use the Levene test based on the median rather the mean. The choice of the number of intervals is somewhat arbitrary, although values of 4 or 8 are reasonable.

$$H_0: \quad \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$$
$$H_a: \quad \text{At least one } \sigma_i^2 \text{ is not equal to the others.}$$

```
Test statistic:  W = 0.09378
Degrees of freedom:  k - 1 = 3
Sample size:  N = 200
Significance level:  α = 0.05
Critical value:  F_{α,k-1,N-k} = 2.651
Critical region:  Reject H0 if W > 2.651
```

In this case, the Levene test indicates that the variances are not significantly different in the four intervals since the test statistic value, 0.9378, is less than the critical value of 2.651.

Randomness

A runs test is used to check for randomness

H_0: the sequence was produced in a random manner
H_a: the sequence was not produced in a random manner

```
Test statistic:  Z = 2.6938
Significance level:  α = 0.05
Critical value:  Z_{1-α/2} = 1.96
Critical region:  Reject H0 if |Z| > 1.96
```

The absolute value of the test statistic is larger than the critical value at the 5 % significance level, so we conclude that the data are not random.

Distributional Assumptions

Since the quantitative tests show that the assumptions of constant scale and non-randomness are not met, the distributional measures will not be meaningful. Therefore these quantitative tests are omitted.

1.4.2.5.3. Develop a Better Model

Sinusoidal Model

The lag plot and autocorrelation plot in the previous section strongly suggested a sinusoidal model might be appropriate. The basic sinusoidal model is:

$$Y_i = C + \alpha \sin(2\pi\omega T_i + \phi) + E_i$$

where C is constant defining a mean level, α is an amplitude for the sine function, ω is the frequency, T_i is a time variable, and ϕ is the phase. This sinusoidal model can be fit using non-linear least squares.

To obtain a good fit, sinusoidal models require good starting values for C, the amplitude, and the frequency.

Good Starting Value for C

A good starting value for C can be obtained by calculating the mean of the data. If the data show a trend, i.e., the assumption of constant location is violated, we can replace C with a linear or quadratic least squares fit. That is, the model becomes

$$Y_i = (B_0 + B_1 * T_i) + \alpha \sin(2\pi\omega T_i + \phi) + E_i$$

or

$$Y_i = (B_0 + B_1 * T_i + B2 * T_i^2) + \alpha \sin(2\pi\omega T_i + \phi) + E_i$$

Since our data did not have any meaningful change of location, we can fit the simpler model with C equal to the mean. From the summary output in the previous page, the mean is -177.44.

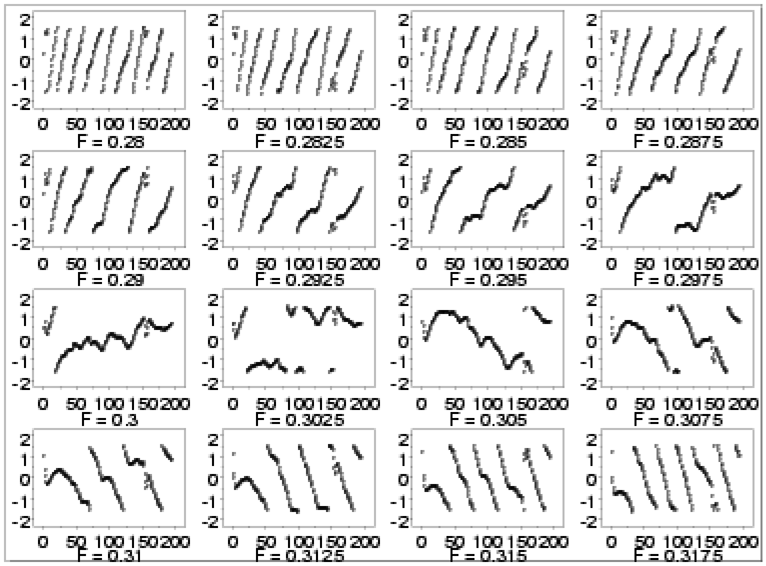
Good Starting Value for Frequency

The starting value for the frequency can be obtained from the spectral plot, which shows the dominant frequency is about 0.3.

Complex Demodulation Phase Plot

The complex demodulation phase plot can be used to refine this initial estimate for the frequency.

For the complex demodulation plot, if the lines slope from left to right, the frequency should be increased. If the lines slope from right to left, it should be decreased. A relatively flat (i.e., horizontal) slope indicates a good frequency. We could generate the demodulation phase plot for 0.3 and then use trial and error to obtain a better estimate for the frequency. To simplify this, we generate 16 of these plots on a single page starting with a frequency of 0.28, increasing in increments of 0.0025, and stopping at 0.3175.



Interpretation

The plots start with lines sloping from left to right but gradually change to a right to left slope. The relatively flat slope occurs for frequency 0.3025 (third row, second column). The complex demodulation phase plot restricts the range from $\pi/2$ to $-\pi/2$. This is why the plot appears to show some breaks.

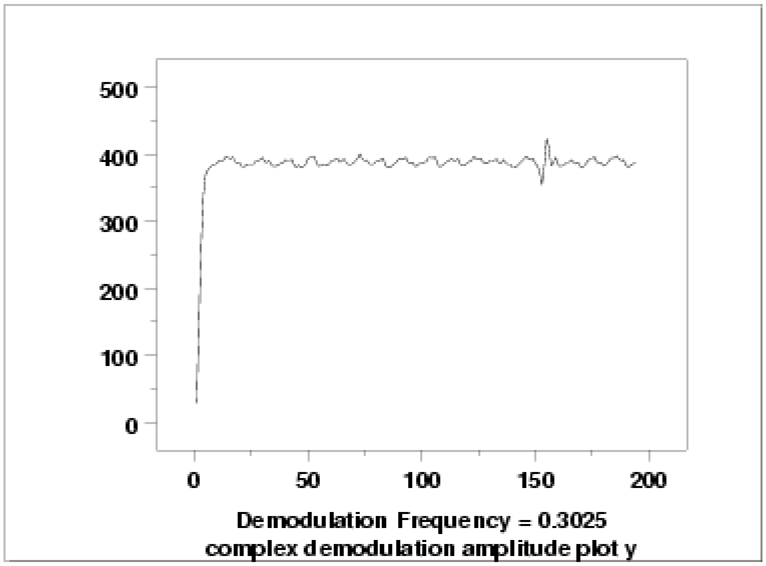
Good Starting Values for Amplitude

The complex demodulation amplitude plot is used to find a good starting value for the amplitude. In addition, this plot indicates whether or not the amplitude is constant over the entire range of the data or if it varies. If the plot is essentially flat, i.e., zero slope, then it is reasonable to assume a constant amplitude in the non-linear model. However, if the slope varies over the range of the plot, we may need to adjust the model to be:

$$Y_i = C + (B_0 + B_1 * T_i)\sin(2\pi\omega T_i + \phi) + E_i$$

That is, we replace α with a function of time. A linear fit is specified in the model above, but this can be replaced with a more elaborate function if needed.

Complex Demodulation Amplitude Plot



The complex demodulation amplitude plot for this data shows that:

1. The amplitude is fixed at approximately 390.
2. There is a short start-up effect.
3. There is a change in amplitude at around x=160 that should be investigated for an outlier.

In terms of a non-linear model, the plot indicates that fitting a single constant for α should be adequate for this data set.

Fit Results

Using starting estimates of 0.3025 for the frequency, 390 for the amplitude, and -177.44 for C, the following parameters were estimated.

```
Coefficient      Estimate      Stan. Error      t-Value
   C             -178.786        11.02           -16.22
   AMP           -361.766        26.19           -13.81
   FREQ          0.302596      0.1510E-03       2005.00
   PHASE          1.46536      0.4909E-01         29.85

Residual Standard Deviation = 155.8484
Residual Degrees of Freedom = 196
```

Model

From the fit results, our proposed model is:

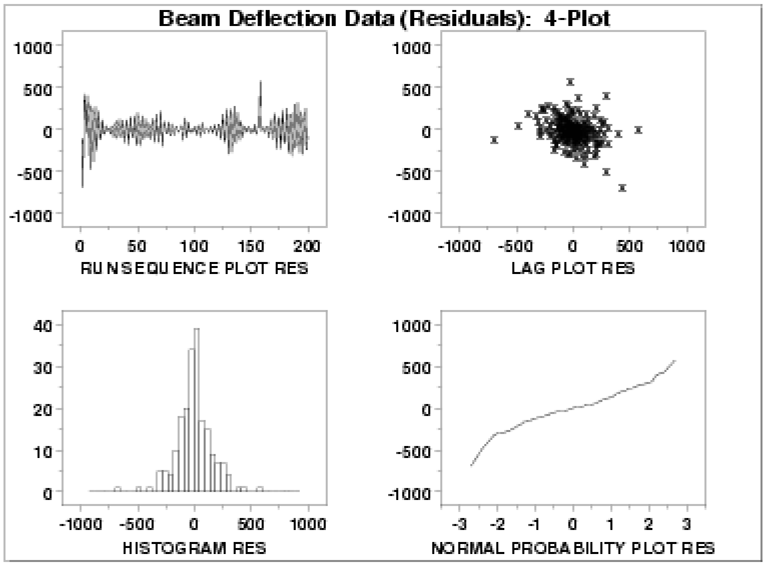
$$\hat{Y}_i = -178.786 - 361.766[2\pi(0.302596)T_i + 1.46536]$$

We will evaluate the adequacy of this model in the next section.

1.4.2.5.4. Validate New Model

4-Plot of Residuals

The first step in evaluating the fit is to generate a 4-plot of the residuals.



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates that the data do not have any significant shifts in location. There does seem to be some shifts in scale. A start-up effect was detected previously by the complex demodulation amplitude plot. There does appear to be a few outliers.

2. The lag plot (upper right) shows that the data are random. The outliers also appear in the lag plot.

3. The histogram (lower left) and the normal probability plot (lower right) do not show any serious non-normality in the residuals. However, the bend in the left portion of the normal probability plot shows some cause for concern.

The 4-plot indicates that this fit is reasonably good. However, we will attempt to improve the fit by removing the outliers.

Fit Results with Outliers Removed

The following parameter estimates were obtained after removing three outliers.

```
Coefficient      Estimate      Stan. Error      t-Value
```

```
C                  -178.788        10.57          -16.91
AMP                -361.759        25.45          -14.22
FREQ               0.302597        0.1457E-03     2077.00
PHASE               1.46533        0.4715E-01       31.08

Residual Standard Deviation = 148.3398
Residual Degrees of Freedom = 193
```

New Fit to Edited Data

The original fit, with a residual standard deviation of 155.84, was:

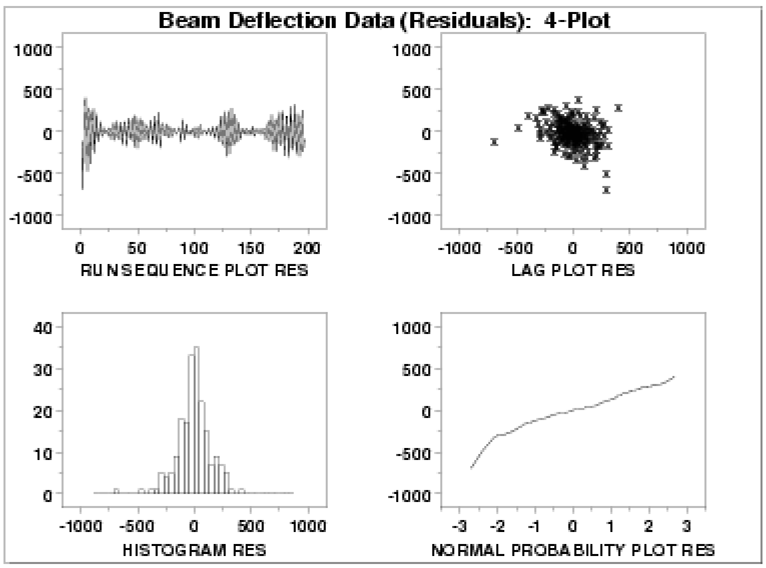
$$\hat{Y}_i = -178.786 - 361.766[2\pi(0.302596)T_i + 1.46536]$$

The new fit, with a residual standard deviation of 148.34, is:

$$\hat{Y}_i = -178.788 - 361.759[2\pi(0.302597)T_i + 1.46533]$$

There is minimal change in the parameter estimates and about a 5 % reduction in the residual standard deviation. In this case, removing the residuals has a modest benefit in terms of reducing the variability of the model.

4-Plot for New Fit



This plot shows that the underlying assumptions are satisfied and therefore the new fit is a good descriptor of the data.

In this case, it is a judgment call whether to use the fit with or without the outliers removed.

1.4.2.5.5. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to repeat the analysis outlined in the case study description on the previous page using Dataplot . It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	**Results and Conclusions**
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step.	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
`1. Invoke Dataplot and read data.` `1. Read in the data.`	`1. You have read 1 column of numbers into Dataplot, variable Y.`

`2. Validate assumptions.`
`1. 4-plot of Y.`
`2. Generate a run sequence plot.`
`3. Generate a lag plot.`
	`1. Based on the 4-plot, there are no obvious shifts in location and scale, but the data are not random.`
	`2. Based on the run sequence plot, there are no obvious shifts in location and`

4. Generate an autocorrelation plot. 5. Generate a spectral plot. 6. Generate a table of summary statistics. 7. Generate a linear fit to detect drift in location. 8. Detect drift in variation by dividing the data into quarters and computing Levene's test statistic for equal standard deviations. 9. Check for randomness by generating a runs test.	scale. 3. Based on the lag plot, the data are not random. 4. The autocorrelation plot shows significant autocorrelation at lag 1. 5. The spectral plot shows a single dominant low frequency peak. 6. The summary statistics table displays 25+ statistics. 7. The linear fit indicates no drift in location since the slope parameter is not statistically significant. 8. Levene's test indicates no significant drift in variation. 9. The runs test indicates significant non-randomness.
3. Fit ***Y_i = C + A*SIN(2*PI*omega*t_i+phi)***. 1. Generate a complex demodulation phase plot. 2. Generate a complex demodulation amplitude plot. 3. Fit the non-linear model.	1. Complex demodulation phase plot indicates a starting frequency of 0.3025. 2. Complex demodulation amplitude plot indicates an amplitude of 390 (but there is a short start-up effect). 3. Non-linear fit generates final parameter estimates. The residual standard deviation from the fit is 155.85 (compared to the standard deviation of 277.73 from the original data).

4. Validate fit. 1. Generate a 4-plot of the residuals from the fit. 2. Generate a nonlinear fit with outliers removed. 3. Generate a 4-plot of the residuals from the fit with the outliers removed.	1. The 4-plot indicates that the assumptions of constant location and scale are valid. The lag plot indicates that the data are random. The histogram and normal probability plot indicate that the residuals that the normality assumption for the residuals are not seriously violated, although there is a bend on the probablity plot that warrants attention. 2. The fit after removing 3 outliers shows some marginal improvement in the model (a 5% reduction in the residual standard deviation). 3. The 4-plot of the model fit after 3 outliers removed shows marginal improvement in satisfying model assumptions.

1. Exploratory Data Analysis
1.4. EDA Case Studies
1.4.2. Case Studies

1.4.2.6. Filter Transmittance

Filter Transmittance

This example illustrates the univariate analysis of filter transmittance data.

1.4.2.6.1. Background and Data

Generation

This data set was collected by NIST chemist Radu Mavrodineaunu in the 1970's from an automatic data acquisition system for a filter transmittance experiment. The response variable is transmittance.

The motivation for studying this data set is to show how the underlying autocorrelation structure in a relatively small data set helped the scientist detect problems with his automatic data acquisition system.

Software

The analyses used in this case study can be generated using both Dataplot code and R code.

Data

The following are the data used for this case study.

```
2.00180
2.00170
2.00180
2.00190
2.00180
2.00170
2.00150
2.00140
2.00150
2.00150
2.00170
2.00180
2.00180
2.00190
2.00190
2.00210
2.00200
2.00160
2.00140
2.00130
2.00130
2.00150
2.00150
2.00160
2.00150
2.00140
2.00130
2.00140
2.00150
2.00140
2.00150
2.00160
2.00150
2.00160
2.00190
2.00200
2.00200
2.00210
2.00220
```

```
2.00230
2.00240
2.00250
2.00270
2.00260
2.00260
2.00260
2.00270
2.00260
2.00250
2.00240
```

NIST SEMATECH HOME TOOLS & AIDS SEARCH BACK NEXT

1.4.2.6.2. Graphical Output and Interpretation

Goal

The goal of this analysis is threefold:

1. Determine if the univariate model:

   $$Y_i = C + E_i$$

   is appropriate and valid.

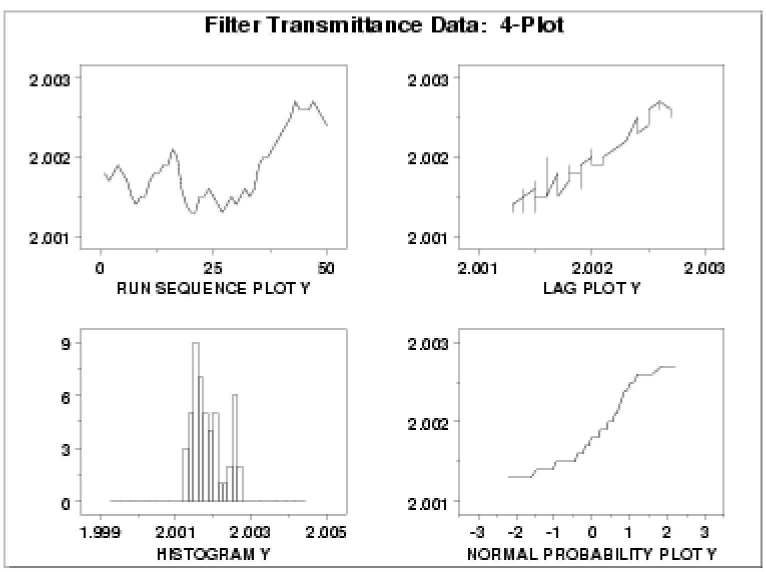
2. Determine if the typical underlying assumptions for an "in control" measurement process are valid. These assumptions are:
   1. random drawings;
   2. from a fixed distribution;
   3. with the distribution having a fixed location; and
   4. the distribution having a fixed scale.

3. Determine if the confidence interval

   $$\bar{Y} \pm 2s/\sqrt{N}$$

   is appropriate and valid where ***s*** is the standard deviation of the original data.

4-Plot of Data



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates a significant shift in location around x=35.

2. The linear appearance in the lag plot (upper right) indicates a non-random pattern in the data.

3. Since the lag plot indicates significant non-randomness, we do not make any interpretation of either the histogram (lower left) or the normal probability plot (lower right).

The serious violation of the non-randomness assumption means that the univariate model

$$Y_i = C + E_i$$

is not valid. Given the linear appearance of the lag plot, the first step might be to consider a model of the type

$$Y_i = A_0 + A_1 * Y_{i-1} + E_i$$

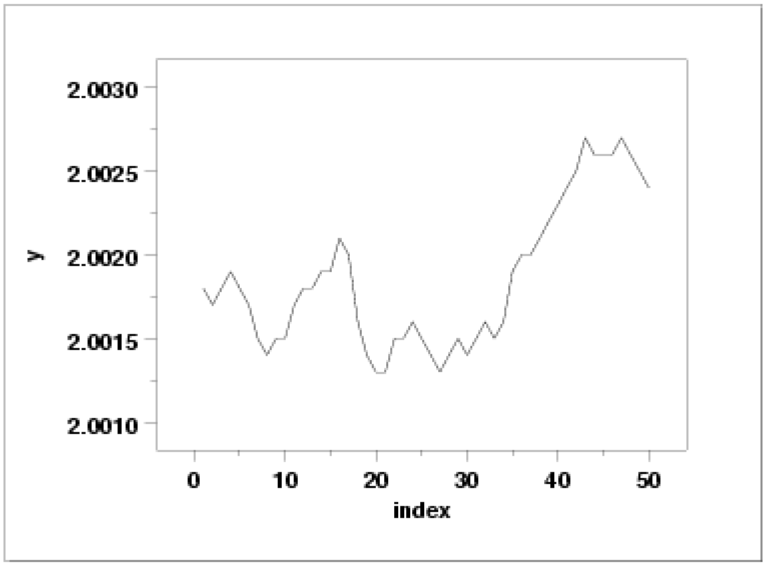
However, in this case discussions with the scientist revealed that non-randomness was entirely unexpected. An examination of the experimental process revealed that the sampling rate for the automatic data acquisition system was too fast. That is, the equipment did not have sufficient time to reset before the next sample started, resulting in the current measurement being contaminated by the previous measurement. The solution was to rerun the experiment allowing more time between samples.

Simple graphical techniques can be quite effective in revealing unexpected results in the data. When this occurs, it is important to investigate whether the unexpected result is due to problems in the experiment and data collection or is indicative of unexpected underlying structure in the data. This determination cannot be made on the basis of statistics alone. The role of the graphical and statistical analysis is to detect problems or unexpected results in the data. Resolving the issues requires the knowledge of the scientist or engineer.

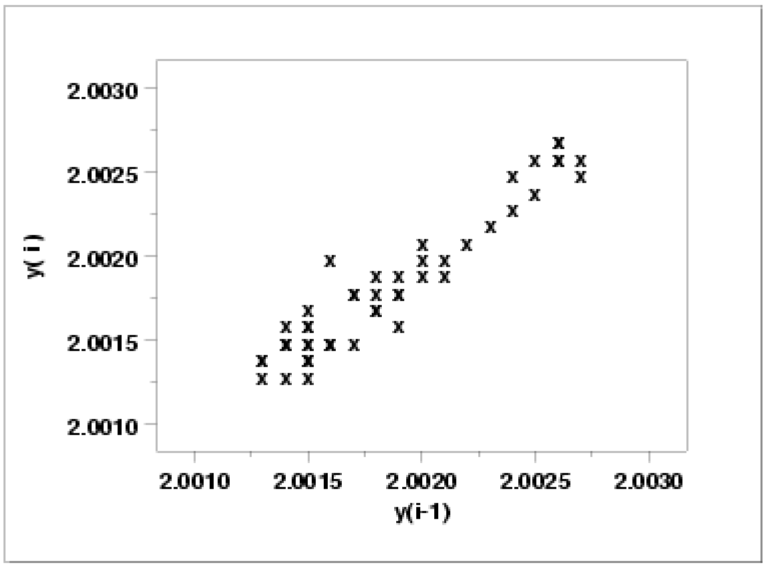
Individual Plots

Although it is generally unnecessary, the plots can be generated individually to give more detail. Since the lag plot indicates significant non-randomness, we omit the distributional plots.

Run Sequence Plot



Lag Plot



NIST SEMATECH | HOME | TOOLS & AIDS | SEARCH | BACK | NEXT

1.4.2.6.3. Quantitative Output and Interpretation

Summary Statistics

As a first step in the analysis, common summary statistics are computed from the data.

```
Sample size  = 50
Mean         =  2.0019
Median       =  2.0018
Minimum      =  2.0013
Maximum      =  2.0027
Range        =  0.0014
Stan. Dev.   =  0.0004
```

Location

One way to quantify a change in location over time is to fit a straight line to the data using an index variable as the independent variable in the regression. For our data, we assume that data are in sequential run order and that the data were collected at equally spaced time intervals. In our regression, we use the index variable X = 1, 2, ..., N, where N is the number of observations. If there is no significant drift in the location over time, the slope parameter should be zero.

Coefficient	Estimate	Stan. Error	*t*-Value
B_0	2.00138	0.9695E-04	0.2064E+05
B_1	0.185E-04	0.3309E-05	5.582

```
Residual Standard Deviation = 0.3376404E-03
Residual Degrees of Freedom = 48
```

The slope parameter, B_1, has a *t* value of 5.582, which is statistically significant. Although the estimated slope, 0.185E-04, is nearly zero, the range of data (2.0013 to 2.0027) is also very small. In this case, we conclude that there is drift in location, although it is relatively small.

Variation

One simple way to detect a change in variation is with a Bartlett test after dividing the data set into several equal sized intervals. However, the Bartlett test is not robust for non-normality. Since the normality assumption is questionable for these data, we use the alternative Levene test. In particular, we use the Levene test based on the median rather the mean. The choice of the number of intervals is somewhat arbitrary, although values of four or eight are reasonable. We will divide our data into four intervals.

H_0: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$
H_a: At least one σ_i^2 is not equal to the others.

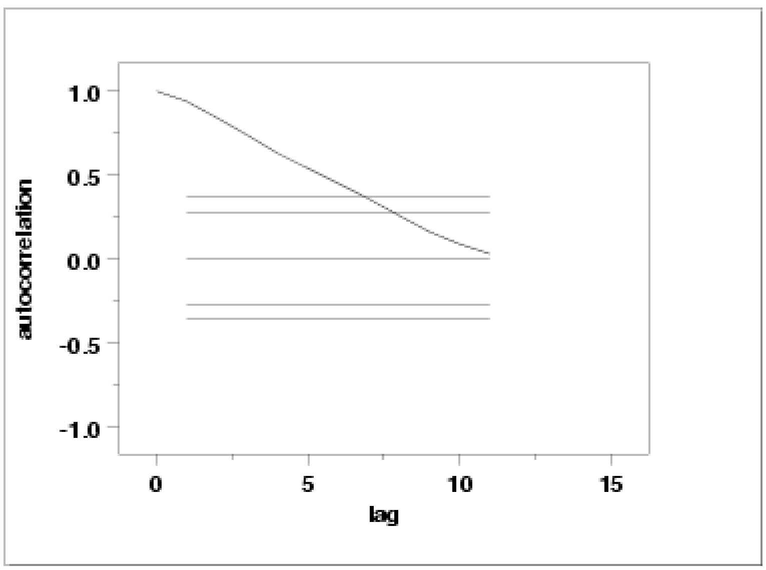
```
Test statistic:   W = 0.971
Degrees of freedom:   k - 1 = 3
Significance level:   α = 0.05
Critical value:  F_{α,k-1,N-k} = 2.806
Critical region:  Reject H0 if W > 2.806
```

In this case, since the Levene test statistic value of 0.971 is less than the critical value of 2.806 at the 5 % level, we conclude that there is no evidence of a change in variation.

Randomness

There are many ways in which data can be non-random. However, most common forms of non-randomness can be detected with a few simple tests. The lag plot in the 4-plot in the previous seciton is a simple graphical technique.

One check is an autocorrelation plot that shows the autocorrelations for various lags. Confidence bands can be plotted at the 95 % and 99 % confidence levels. Points outside this band indicate statistically significant values (lag 0 is always 1).



The lag 1 autocorrelation, which is generally the one of most interest, is 0.93. The critical values at the 5 % level are -0.277 and 0.277. This indicates that the lag 1 autocorrelation is statistically significant, so there is strong evidence of non-randomness.

A common test for randomness is the runs test.

H_0: the sequence was produced in a random manner
H_a: the sequence was not produced in a random manner

Test statistic: $Z = -5.3246$

```
Significance level:  α = 0.05
Critical value:  Z1-α/2 = 1.96
Critical region:  Reject H0 if |Z| > 1.96
```

Because the test statistic is outside of the critical region, we reject the null hypothesis and conclude that the data are not random.

Distributional Analysis

Since we rejected the randomness assumption, the distributional tests are not meaningful. Therefore, these quantitative tests are omitted. We also omit Grubbs' outlier test since it also assumes the data are approximately normally distributed.

Univariate Report

It is sometimes useful and convenient to summarize the above results in a report.

```
 Analysis for filter transmittance data

 1: Sample Size                                = 50

 2: Location
    Mean                                        =
2.001857
    Standard Deviation of Mean                 =
0.00006
    95% Confidence Interval for Mean           =
(2.001735,2.001979)
    Drift with respect to location?            = NO

 3: Variation
    Standard Deviation                         =
0.00043
    95% Confidence Interval for SD             =
(0.000359,0.000535)
    Change in variation?
    (based on Levene's test on quarters
    of the data)                                = NO

 4: Distribution
    Distributional tests omitted due to
    non-randomness of the data

 5: Randomness
    Lag One Autocorrelation                    =
0.937998
    Data are Random?
      (as measured by autocorrelation)        = NO

 6: Statistical Control
    (i.e., no drift in location or scale,
    data are random, distribution is
    fixed, here we are testing only for
    normal)
    Data Set is in Statistical Control?        = NO

 7: Outliers?
    (Grubbs' test omitted)                     = NO
```

1.4.2.6.4. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to repeat the analysis outlined in the case study description on the previous page using Dataplot . It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	**Results and Conclusions**
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step.	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
`1. Invoke Dataplot and read data.` `1. Read in the data.`	`1. You have read 1 column of numbers into Dataplot, variable Y.`
`2. 4-plot of the data.` `1. 4-plot of Y.`	`1. Based on the 4-plot, there is a shift in location and the data are not random.`
`3. Generate the individual plots.` `1. Generate a run sequence plot.`	`1. The run sequence`

2. Generate a lag plot.	plot indicates that there is a shift in location. 2. The strong linear pattern of the lag plot indicates significant non-randomness.
4. Generate summary statistics, quantitative analysis, and print a univariate report. 1. Generate a table of summary statistics. 2. Compute a linear fit based on quarters of the data to detect drift in location. 3. Compute Levene's test based on quarters of the data to detect changes in variation. 4. Check for randomness by generating an autocorrelation plot and a runs test. 5. Print a univariate report (this assumes steps 2 thru 4 have already been run).	1. The summary statistics table displays 25+ statistics. 2. The linear fit indicates a slight drift in location since the slope parameter is statistically significant, but small. 3. Levene's test indicates no significant drift in variation. 4. The lag 1 autocorrelation is 0.94. This is outside the 95% confidence interval bands which indicates significant non-randomness. 5. The results are summarized in a convenient report.

1. Exploratory Data Analysis
1.4. EDA Case Studies
1.4.2. Case Studies

1.4.2.7. Standard Resistor

Standard Resistor

This example illustrates the univariate analysis of standard resistor data.

1.4.2.7.1. Background and Data

Generation

This data set was collected by Ron Dziuba of NIST over a 5-year period from 1980 to 1985. The response variable is resistor values.

The motivation for studying this data set is to illustrate data that violate the assumptions of constant location and scale.

Software

The analyses used in this case study can be generated using both Dataplot code and R code.

Data

The following are the data used for this case study.

```
27.8680
27.8929
27.8773
27.8530
27.8876
27.8725
27.8743
27.8879
27.8728
27.8746
27.8863
27.8716
27.8818
27.8872
27.8885
27.8945
27.8797
27.8627
27.8870
27.8895
27.9138
27.8931
27.8852
27.8788
27.8827
27.8939
27.8558
27.8814
27.8479
27.8479
27.8848
27.8809
27.8479
27.8611
27.8630
27.8679
27.8637
27.8985
27.8900
27.8577
27.8848
27.8869
27.8976
```

```
27.8610
27.8567
27.8417
27.8280
27.8555
27.8639
27.8702
27.8582
27.8605
27.8900
27.8758
27.8774
27.9008
27.8988
27.8897
27.8990
27.8958
27.8830
27.8967
27.9105
27.9028
27.8977
27.8953
27.8970
27.9190
27.9180
27.8997
27.9204
27.9234
27.9072
27.9152
27.9091
27.8882
27.9035
27.9267
27.9138
27.8955
27.9203
27.9239
27.9199
27.9646
27.9411
27.9345
27.8712
27.9145
27.9259
27.9317
27.9239
27.9247
27.9150
27.9444
27.9457
27.9166
27.9066
27.9088
27.9255
27.9312
27.9439
27.9210
27.9102
27.9083
27.9121
27.9113
27.9091
27.9235
27.9291
27.9253
27.9092
27.9117
27.9194
27.9039
27.9515
27.9143
27.9124
27.9128
27.9260
27.9339
27.9500
27.9530
27.9430
27.9400
```

```
27.8850
27.9350
27.9120
27.9260
27.9660
27.9280
27.9450
27.9390
27.9429
27.9207
27.9205
27.9204
27.9198
27.9246
27.9366
27.9234
27.9125
27.9032
27.9285
27.9561
27.9616
27.9530
27.9280
27.9060
27.9380
27.9310
27.9347
27.9339
27.9410
27.9397
27.9472
27.9235
27.9315
27.9368
27.9403
27.9529
27.9263
27.9347
27.9371
27.9129
27.9549
27.9422
27.9423
27.9750
27.9339
27.9629
27.9587
27.9503
27.9573
27.9518
27.9527
27.9589
27.9300
27.9629
27.9630
27.9660
27.9730
27.9660
27.9630
27.9570
27.9650
27.9520
27.9820
27.9560
27.9670
27.9520
27.9470
27.9720
27.9610
27.9437
27.9660
27.9580
27.9660
27.9700
27.9600
27.9660
27.9770
27.9110
27.9690
27.9698
27.9616
```

```
27.9371
27.9700
27.9265
27.9964
27.9842
27.9667
27.9610
27.9943
27.9616
27.9397
27.9799
28.0086
27.9709
27.9741
27.9675
27.9826
27.9676
27.9703
27.9789
27.9786
27.9722
27.9831
28.0043
27.9548
27.9875
27.9495
27.9549
27.9469
27.9744
27.9744
27.9449
27.9837
27.9585
28.0096
27.9762
27.9641
27.9854
27.9877
27.9839
27.9817
27.9845
27.9877
27.9880
27.9822
27.9836
28.0030
27.9678
28.0146
27.9945
27.9805
27.9785
27.9791
27.9817
27.9805
27.9782
27.9753
27.9792
27.9704
27.9794
27.9814
27.9794
27.9795
27.9881
27.9772
27.9796
27.9736
27.9772
27.9960
27.9795
27.9779
27.9829
27.9829
27.9815
27.9811
27.9773
27.9778
27.9724
27.9756
27.9699
27.9724
27.9666
```

```
27.9666
27.9739
27.9684
27.9861
27.9901
27.9879
27.9865
27.9876
27.9814
27.9842
27.9868
27.9834
27.9892
27.9864
27.9843
27.9838
27.9847
27.9860
27.9872
27.9869
27.9602
27.9852
27.9860
27.9836
27.9813
27.9623
27.9843
27.9802
27.9863
27.9813
27.9881
27.9850
27.9850
27.9830
27.9866
27.9888
27.9841
27.9863
27.9903
27.9961
27.9905
27.9945
27.9878
27.9929
27.9914
27.9914
27.9997
28.0006
27.9999
28.0004
28.0020
28.0029
28.0008
28.0040
28.0078
28.0065
27.9959
28.0073
28.0017
28.0042
28.0036
28.0055
28.0007
28.0066
28.0011
27.9960
28.0083
27.9978
28.0108
28.0088
28.0088
28.0139
28.0092
28.0092
28.0049
28.0111
28.0120
28.0093
28.0116
28.0102
28.0139
```

```
28.0113
28.0158
28.0156
28.0137
28.0236
28.0171
28.0224
28.0184
28.0199
28.0190
28.0204
28.0170
28.0183
28.0201
28.0182
28.0183
28.0175
28.0127
28.0211
28.0057
28.0180
28.0183
28.0149
28.0185
28.0182
28.0192
28.0213
28.0216
28.0169
28.0162
28.0167
28.0167
28.0169
28.0169
28.0161
28.0152
28.0179
28.0215
28.0194
28.0115
28.0174
28.0178
28.0202
28.0240
28.0198
28.0194
28.0171
28.0134
28.0121
28.0121
28.0141
28.0101
28.0114
28.0122
28.0124
28.0171
28.0165
28.0166
28.0159
28.0181
28.0200
28.0116
28.0144
28.0141
28.0116
28.0107
28.0169
28.0105
28.0136
28.0138
28.0114
28.0122
28.0122
28.0116
28.0025
28.0097
28.0066
28.0072
28.0066
28.0068
28.0067
```

```
28.0130
28.0091
28.0088
28.0091
28.0091
28.0115
28.0087
28.0128
28.0139
28.0095
28.0115
28.0101
28.0121
28.0114
28.0121
28.0122
28.0121
28.0168
28.0212
28.0219
28.0221
28.0204
28.0169
28.0141
28.0142
28.0147
28.0159
28.0165
28.0144
28.0182
28.0155
28.0155
28.0192
28.0204
28.0185
28.0248
28.0185
28.0226
28.0271
28.0290
28.0240
28.0302
28.0243
28.0288
28.0287
28.0301
28.0273
28.0313
28.0293
28.0300
28.0344
28.0308
28.0291
28.0287
28.0358
28.0309
28.0286
28.0308
28.0291
28.0380
28.0411
28.0420
28.0359
28.0368
28.0327
28.0361
28.0334
28.0300
28.0347
28.0359
28.0344
28.0370
28.0355
28.0371
28.0318
28.0390
28.0390
28.0390
28.0376
28.0376
28.0377
```

```
28.0345
28.0333
28.0429
28.0379
28.0401
28.0401
28.0423
28.0393
28.0382
28.0424
28.0386
28.0386
28.0373
28.0397
28.0412
28.0565
28.0419
28.0456
28.0426
28.0423
28.0391
28.0403
28.0388
28.0408
28.0457
28.0455
28.0460
28.0456
28.0464
28.0442
28.0416
28.0451
28.0432
28.0434
28.0448
28.0448
28.0373
28.0429
28.0392
28.0469
28.0443
28.0356
28.0474
28.0446
28.0348
28.0368
28.0418
28.0445
28.0533
28.0439
28.0474
28.0435
28.0419
28.0538
28.0538
28.0463
28.0491
28.0441
28.0411
28.0507
28.0459
28.0519
28.0554
28.0512
28.0507
28.0582
28.0471
28.0539
28.0530
28.0502
28.0422
28.0431
28.0395
28.0177
28.0425
28.0484
28.0693
28.0490
28.0453
28.0494
28.0522
```

```
28.0393
28.0443
28.0465
28.0450
28.0539
28.0566
28.0585
28.0486
28.0427
28.0548
28.0616
28.0298
28.0726
28.0695
28.0629
28.0503
28.0493
28.0537
28.0613
28.0643
28.0678
28.0564
28.0703
28.0647
28.0579
28.0630
28.0716
28.0586
28.0607
28.0601
28.0611
28.0606
28.0611
28.0066
28.0412
28.0558
28.0590
28.0750
28.0483
28.0599
28.0490
28.0499
28.0565
28.0612
28.0634
28.0627
28.0519
28.0551
28.0696
28.0581
28.0568
28.0572
28.0529
28.0421
28.0432
28.0211
28.0363
28.0436
28.0619
28.0573
28.0499
28.0340
28.0474
28.0534
28.0589
28.0466
28.0448
28.0576
28.0558
28.0522
28.0480
28.0444
28.0429
28.0624
28.0610
28.0461
28.0564
28.0734
28.0565
28.0503
28.0581
```

```
28.0519
28.0625
28.0583
28.0645
28.0642
28.0535
28.0510
28.0542
28.0677
28.0416
28.0676
28.0596
28.0635
28.0558
28.0623
28.0718
28.0585
28.0552
28.0684
28.0646
28.0590
28.0465
28.0594
28.0303
28.0533
28.0561
28.0585
28.0497
28.0582
28.0507
28.0562
28.0715
28.0468
28.0411
28.0587
28.0456
28.0705
28.0534
28.0558
28.0536
28.0552
28.0461
28.0598
28.0598
28.0650
28.0423
28.0442
28.0449
28.0660
28.0506
28.0655
28.0512
28.0407
28.0475
28.0411
28.0512
28.1036
28.0641
28.0572
28.0700
28.0577
28.0637
28.0534
28.0461
28.0701
28.0631
28.0575
28.0444
28.0592
28.0684
28.0593
28.0677
28.0512
28.0644
28.0660
28.0542
28.0768
28.0515
28.0579
28.0538
28.0526
```

```
28.0833
28.0637
28.0529
28.0535
28.0561
28.0736
28.0635
28.0600
28.0520
28.0695
28.0608
28.0608
28.0590
28.0290
28.0939
28.0618
28.0551
28.0757
28.0698
28.0717
28.0529
28.0644
28.0613
28.0759
28.0745
28.0736
28.0611
28.0732
28.0782
28.0682
28.0756
28.0857
28.0739
28.0840
28.0862
28.0724
28.0727
28.0752
28.0732
28.0703
28.0849
28.0795
28.0902
28.0874
28.0971
28.0638
28.0877
28.0751
28.0904
28.0971
28.0661
28.0711
28.0754
28.0516
28.0961
28.0689
28.1110
28.1062
28.0726
28.1141
28.0913
28.0982
28.0703
28.0654
28.0760
28.0727
28.0850
28.0877
28.0967
28.1185
28.0945
28.0834
28.0764
28.1129
28.0797
28.0707
28.1008
28.0971
28.0826
28.0857
28.0984
```

```
28.0869
28.0795
28.0875
28.1184
28.0746
28.0816
28.0879
28.0888
28.0924
28.0979
28.0702
28.0847
28.0917
28.0834
28.0823
28.0917
28.0779
28.0852
28.0863
28.0942
28.0801
28.0817
28.0922
28.0914
28.0868
28.0832
28.0881
28.0910
28.0886
28.0961
28.0857
28.0859
28.1086
28.0838
28.0921
28.0945
28.0839
28.0877
28.0803
28.0928
28.0885
28.0940
28.0856
28.0849
28.0955
28.0955
28.0846
28.0871
28.0872
28.0917
28.0931
28.0865
28.0900
28.0915
28.0963
28.0917
28.0950
28.0898
28.0902
28.0867
28.0843
28.0939
28.0902
28.0911
28.0909
28.0949
28.0867
28.0932
28.0891
28.0932
28.0887
28.0925
28.0928
28.0883
28.0946
28.0977
28.0914
28.0959
28.0926
28.0923
28.0950
```

```
28.1006
28.0924
28.0963
28.0893
28.0956
28.0980
28.0928
28.0951
28.0958
28.0912
28.0990
28.0915
28.0957
28.0976
28.0888
28.0928
28.0910
28.0902
28.0950
28.0995
28.0965
28.0972
28.0963
28.0946
28.0942
28.0998
28.0911
28.1043
28.1002
28.0991
28.0959
28.0996
28.0926
28.1002
28.0961
28.0983
28.0997
28.0959
28.0988
28.1029
28.0989
28.1000
28.0944
28.0979
28.1005
28.1012
28.1013
28.0999
28.0991
28.1059
28.0961
28.0981
28.1045
28.1047
28.1042
28.1146
28.1113
28.1051
28.1065
28.1065
28.0985
28.1000
28.1066
28.1041
28.0954
28.1090
```

1.4.2.7.2. Graphical Output and Interpretation

Goal

The goal of this analysis is threefold:

1. Determine if the univariate model:

   $$Y_i = C + E_i$$

   is appropriate and valid.

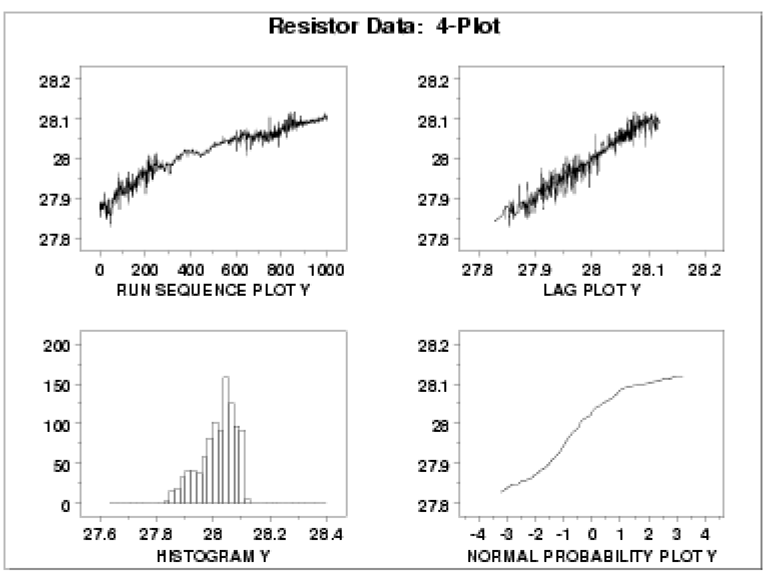
2. Determine if the typical underlying assumptions for an "in control" measurement process are valid. These assumptions are:
   1. random drawings;
   2. from a fixed distribution;
   3. with the distribution having a fixed location; and
   4. the distribution having a fixed scale.

3. Determine if the confidence interval

   $$\bar{Y} \pm 2s/\sqrt{N}$$

   is appropriate and valid where ***s*** is the standard deviation of the original data.

4-Plot of Data



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates significant shifts in both location and variation. Specifically, the location is increasing with time. The variability seems greater in the first and last third of the data than it does in the middle third.

2. The lag plot (upper right) shows a significant non-random pattern in the data. Specifically, the strong linear appearance of this plot is indicative of a model that relates Y_t to Y_{t-1}.

3. The distributional plots, the histogram (lower left) and the normal probability plot (lower right), are not interpreted since the randomness assumption is so clearly violated.

The serious violation of the non-randomness assumption means that the univariate model

$$Y_i = C + E_i$$

is not valid. Given the linear appearance of the lag plot, the first step might be to consider a model of the type

$$Y_i = A_0 + A_1 * Y_{i-1} + E_i$$

However, discussions with the scientist revealed the following:

1. the drift with respect to location was expected.

2. the non-constant variability was not expected.

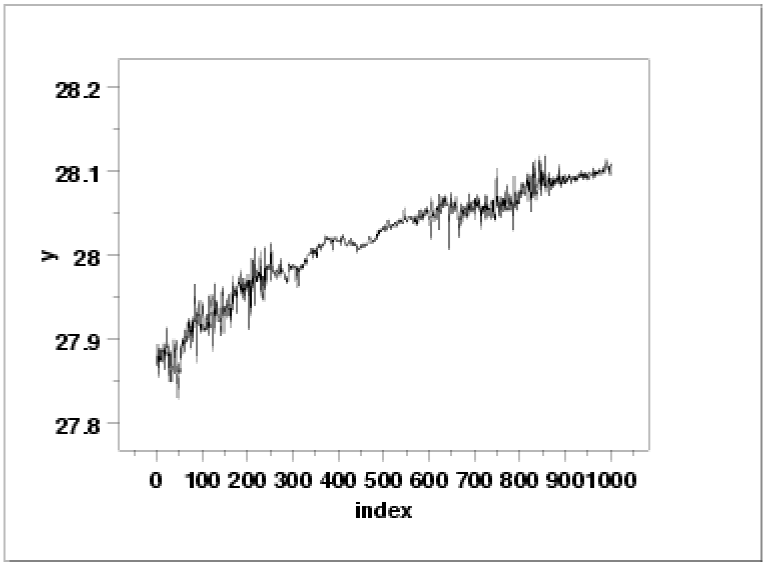
The scientist examined the data collection device and determined that the non-constant variation was a seasonal effect. The high variability data in the first and last thirds was collected in winter while the more stable middle third was collected in the summer. The seasonal effect was determined to be caused by the amount of humidity affecting the measurement equipment. In this case, the solution was to modify the test equipment to be less sensitive to enviromental factors.

Simple graphical techniques can be quite effective in revealing unexpected results in the data. When this occurs, it is important to investigate whether the unexpected result is due to problems in the experiment and data collection, or is it in fact indicative of an unexpected underlying structure in the data. This determination cannot be made on the basis of statistics alone. The role of the graphical and statistical analysis is to detect problems or unexpected results in the data. Resolving the issues requires the knowledge of the scientist or engineer.

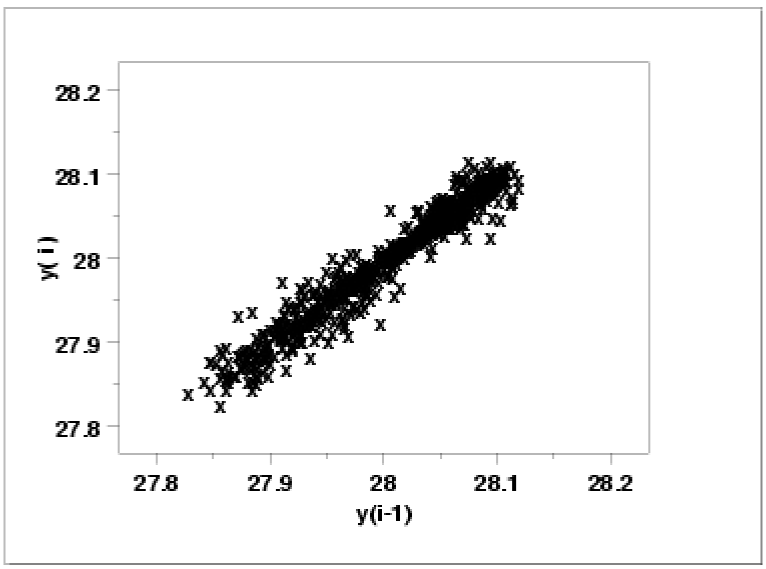
Individual Plots

Although it is generally unnecessary, the plots can be generated individually to give more detail. Since the lag plot indicates significant non-randomness, we omit the distributional plots.

Run Sequence Plot



Lag Plot



1.4.2.7.3. Quantitative Output and Interpretation

Summary Statistics

As a first step in the analysis, common summary statistics are computed from the data.

```
Sample size  = 1000
Mean         =   28.01634
Median       =   28.02910
Minimum      =   27.82800
Maximum      =   28.11850
Range        =    0.29050
Stan. Dev.   =    0.06349
```

Location

One way to quantify a change in location over time is to fit a straight line to the data using an index variable as the independent variable in the regression. For our data, we assume that data are in sequential run order and that the data were collected at equally spaced time intervals. In our regression, we use the index variable X = 1, 2, ..., N, where N is the number of observations. If there is no significant drift in the location over time, the slope parameter should be zero.

Coefficient	Estimate	Stan. Error	*t*-Value
B_0	27.9114	0.1209E-02	0.2309E+05
B_1	0.20967E-03	0.2092E-05	100.2

```
Residual Standard Deviation = 0.1909796E-01
Residual Degrees of Freedom = 998
```

The slope parameter, B_1, has a *t* value of 100.2 which is statistically significant. The value of the slope parameter estimate is 0.00021. Although this number is nearly zero, we need to take into account that the original scale of the data is from about 27.8 to 28.2. In this case, we conclude that there is a drift in location.

Variation

One simple way to detect a change in variation is with a Bartlett test after dividing the data set into several equal-sized intervals. However, the Bartlett test is not robust for non-normality. Since the normality assumption is questionable for these data, we use the alternative Levene test. In particular, we use the Levene test based on the median rather the mean. The choice of the number of intervals is somewhat arbitrary, although values of four or eight are reasonable. We will divide our data into four intervals.

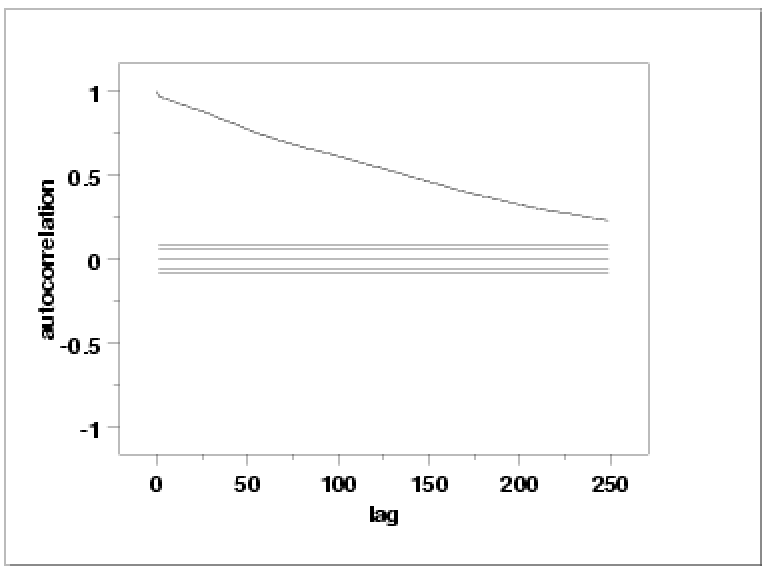
```
      H0:  σ1² = σ2² = σ3² = σ4²
      Ha:  At least one σi² is not equal to the others.

      Test statistic:  W = 140.85
      Degrees of freedom:  k - 1 = 3
      Significance level:  α = 0.05
      Critical value:  Fα,k-1,N-k = 2.614
      Critical region:  Reject H0 if W > 2.614
```

In this case, since the Levene test statistic value of 140.85 is greater than the 5 % significance level critical value of 2.614, we conclude that there is significant evidence of nonconstant variation.

Randomness

There are many ways in which data can be non-random. However, most common forms of non-randomness can be detected with a few simple tests. The lag plot in the 4-plot in the previous section is a simple graphical technique.

One check is an autocorrelation plot that shows the autocorrelations for various lags. Confidence bands can be plotted at the 95 % and 99 % confidence levels. Points outside this band indicate statistically significant values (lag 0 is always 1).



The lag 1 autocorrelation, which is generally the one of greatest interest, is 0.97. The critical values at the 5 % significance level are -0.062 and 0.062. This indicates that the lag 1 autocorrelation is statistically significant, so there is strong evidence of non-randomness.

A common test for randomness is the runs test.

```
      H0:  the sequence was produced in a random manner
      Ha:  the sequence was not produced in a
```

```
random manner

      Test statistic:  Z = -30.5629
      Significance level:  α = 0.05
      Critical value:  Z_{1-α/2} = 1.96
      Critical region:  Reject H0 if |Z| > 1.96
```

Because the test statistic is outside of the critical region, we reject the null hypothesis and conclude that the data are not random.

Distributional Analysis

Since we rejected the randomness assumption, the distributional tests are not meaningful. Therefore, these quantitative tests are omitted. Since the Grubbs' test for outliers also assumes the approximate normality of the data, we omit Grubbs' test as well.

Univariate Report

It is sometimes useful and convenient to summarize the above results in a report.

```
 Analysis for resistor case study

 1: Sample Size                                = 1000

 2: Location
    Mean                                        =
28.01635
    Standard Deviation of Mean                 =
0.002008
    95% Confidence Interval for Mean           =
(28.0124,28.02029)
    Drift with respect to location?            = NO

 3: Variation
    Standard Deviation                         =
0.063495
    95% Confidence Interval for SD             =
(0.060829,0.066407)
    Change in variation?
    (based on Levene's test on quarters
    of the data)                                = YES

 4: Randomness
    Autocorrelation                            =
0.972158
    Data Are Random?
      (as measured by autocorrelation)        = NO

 5: Distribution
    Distributional test omitted due to
    non-randomness of the data

 6: Statistical Control
    (i.e., no drift in location or scale,
    data are random, distribution is
    fixed)
    Data Set is in Statistical Control?       = NO

 7: Outliers?
    (Grubbs' test omitted due to
    non-randomness of the data)
```

1.4.2.7.4. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to repeat the analysis outlined in the case study description on the previous page using Dataplot . It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	Results and Conclusions
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step. *NOTE: This case study has 1,000 points. For better performance, it is highly recommended that you check the "No Update" box on the Spreadsheet window for this case study. This will suppress subsequent updating of the Spreadsheet window as the data are created or modified.*	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
`1. Invoke Dataplot and read data.` `1. Read in the data.`	`1. You have read 1 column of numbers into Dataplot, variable Y.`
`2. 4-plot of the data.` `1. 4-plot of Y.`	`1. Based on the 4-plot, there are shifts`

	in location and variation and the data are not random.
3. Generate the individual plots. 1. Generate a run sequence plot. 2. Generate a lag plot.	1. The run sequence plot indicates that there are shifts of location and variation. 2. The lag plot shows a strong linear pattern, which indicates significant non-randomness.
4. Generate summary statistics, quantitative analysis, and print a univariate report. 1. Generate a table of summary statistics. 2. Generate the sample mean, a confidence interval for the population mean, and compute a linear fit to detect drift in location. 3. Generate the sample standard deviation, a confidence interval for the population standard deviation, and detect drift in variation by dividing the data into quarters and computing Levene's test for equal standard deviations. 4. Check for randomness by generating an autocorrelation plot and a runs test. 5. Print a univariate report (this assumes steps 2 thru 5 have already been run).	1. The summary statistics table displays 25+ statistics. 2. The mean is 28.0163 and a 95% confidence interval is (28.0124,28.02029). The linear fit indicates drift in location since the slope parameter estimate is statistically significant. 3. The standard deviation is 0.0635 with a 95% confidence interval of (0.060829,0.066407). Levene's test indicates significant change in variation. 4. The lag 1 autocorrelation is 0.97. From the autocorrelation plot, this is outside the 95% confidence interval bands, indicating significant non-randomness. 5. The results are

	summarized in a convenient report.

1.4.2.8. Heat Flow Meter 1

Heat Flow Meter Calibration and Stability

This example illustrates the univariate analysis of standard resistor data.

1.4.2.8.1. Background and Data

Generation

This data set was collected by Bob Zarr of NIST in January, 1990 from a heat flow meter calibration and stability analysis. The response variable is a calibration factor.

The motivation for studying this data set is to illustrate a well-behaved process where the underlying assumptions hold and the process is in statistical control.

Software

The analyses used in this case study can be generated using both Dataplot code and R code.

Data

The following are the data used for this case study.

```
9.206343
9.299992
9.277895
9.305795
9.275351
9.288729
9.287239
9.260973
9.303111
9.275674
9.272561
9.288454
9.255672
9.252141
9.297670
9.266534
9.256689
9.277542
9.248205
9.252107
9.276345
9.278694
9.267144
9.246132
9.238479
9.269058
9.248239
9.257439
9.268481
9.288454
9.258452
9.286130
9.251479
9.257405
9.268343
9.291302
9.219460
9.270386
9.218808
9.241185
9.269989
9.226585
```

```
9.258556
9.286184
9.320067
9.327973
9.262963
9.248181
9.238644
9.225073
9.220878
9.271318
9.252072
9.281186
9.270624
9.294771
9.301821
9.278849
9.236680
9.233988
9.244687
9.221601
9.207325
9.258776
9.275708
9.268955
9.257269
9.264979
9.295500
9.292883
9.264188
9.280731
9.267336
9.300566
9.253089
9.261376
9.238409
9.225073
9.235526
9.239510
9.264487
9.244242
9.277542
9.310506
9.261594
9.259791
9.253089
9.245735
9.284058
9.251122
9.275385
9.254619
9.279526
9.275065
9.261952
9.275351
9.252433
9.230263
9.255150
9.268780
9.290389
9.274161
9.255707
9.261663
9.250455
9.261952
9.264041
9.264509
9.242114
9.239674
9.221553
9.241935
9.215265
9.285930
9.271559
9.266046
9.285299
9.268989
9.267987
9.246166
9.231304
9.240768
9.260506
```

```
9.274355
9.292376
9.271170
9.267018
9.308838
9.264153
9.278822
9.255244
9.229221
9.253158
9.256292
9.262602
9.219793
9.258452
9.267987
9.267987
9.248903
9.235153
9.242933
9.253453
9.262671
9.242536
9.260803
9.259825
9.253123
9.240803
9.238712
9.263676
9.243002
9.246826
9.252107
9.261663
9.247311
9.306055
9.237646
9.248937
9.256689
9.265777
9.299047
9.244814
9.287205
9.300566
9.256621
9.271318
9.275154
9.281834
9.253158
9.269024
9.282077
9.277507
9.284910
9.239840
9.268344
9.247778
9.225039
9.230750
9.270024
9.265095
9.284308
9.280697
9.263032
9.291851
9.252072
9.244031
9.283269
9.196848
9.231372
9.232963
9.234956
9.216746
9.274107
9.273776
```

1. Exploratory Data Analysis
1.4. EDA Case Studies
1.4.2. Case Studies
1.4.2.8. Heat Flow Meter 1

1.4.2.8.2. Graphical Output and Interpretation

Goal

The goal of this analysis is threefold:

1. Determine if the univariate model:

   $$Y_i = C + E_i$$

   is appropriate and valid.

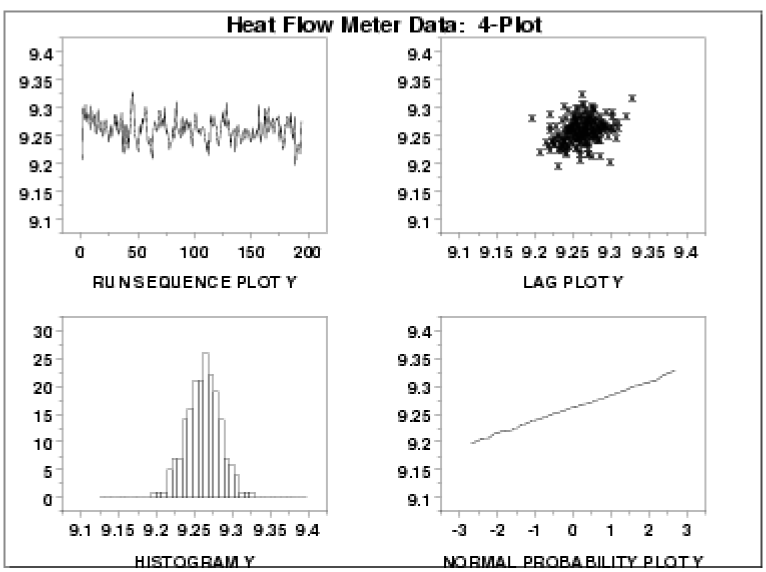
2. Determine if the typical underlying assumptions for an "in control" measurement process are valid. These assumptions are:
   1. random drawings;
   2. from a fixed distribution;
   3. with the distribution having a fixed location; and
   4. the distribution having a fixed scale.

3. Determine if the confidence interval

   $$\bar{Y} \pm 2s/\sqrt{N}$$

   is appropriate and valid where ***s*** is the standard deviation of the original data.

4-Plot of Data



Interpretation

The assumptions are addressed by the graphics shown above:

1. The run sequence plot (upper left) indicates that the data do not have any significant shifts in location or scale over time.

2. The lag plot (upper right) does not indicate any non-random pattern in the data.

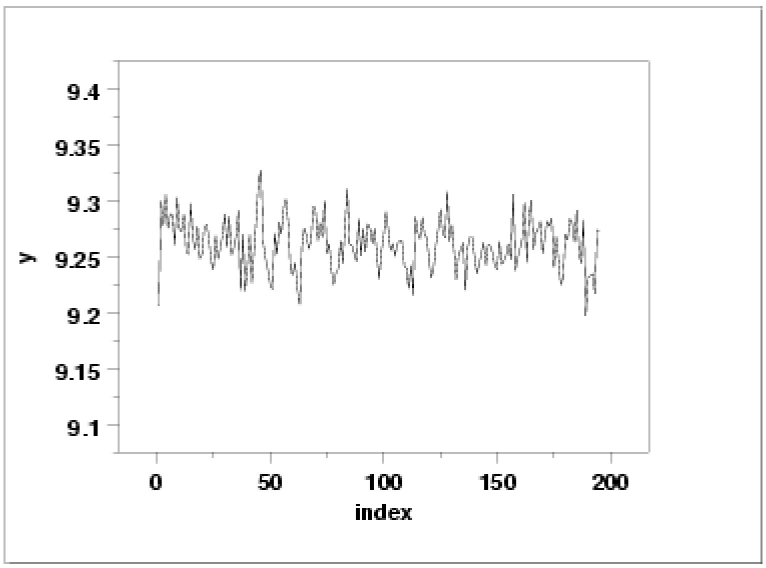
3. The histogram (lower left) shows that the data are reasonably symmetric, there does not appear to be significant outliers in the tails, and it seems reasonable to assume that the data are from approximately a normal distribution.

4. The normal probability plot (lower right) verifies that an assumption of normality is in fact reasonable.

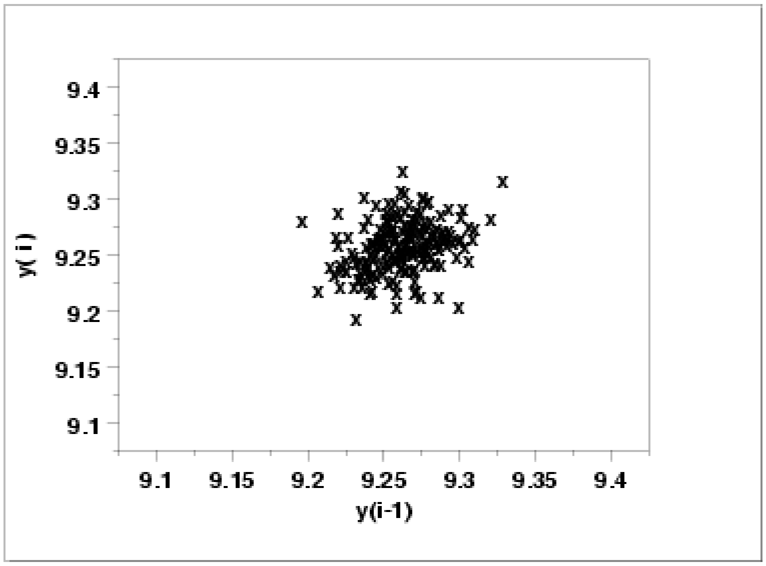
Individual Plots

Although it is generally unnecessary, the plots can be generated individually to give more detail.

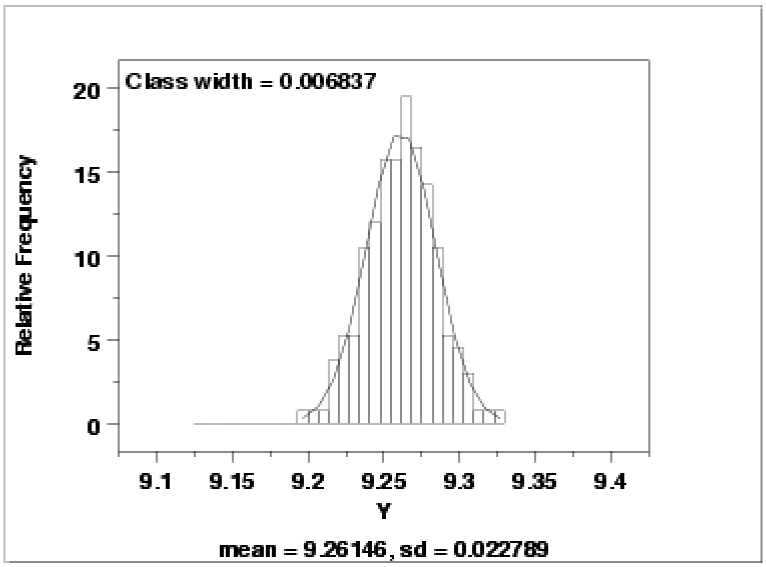
Run Sequence Plot



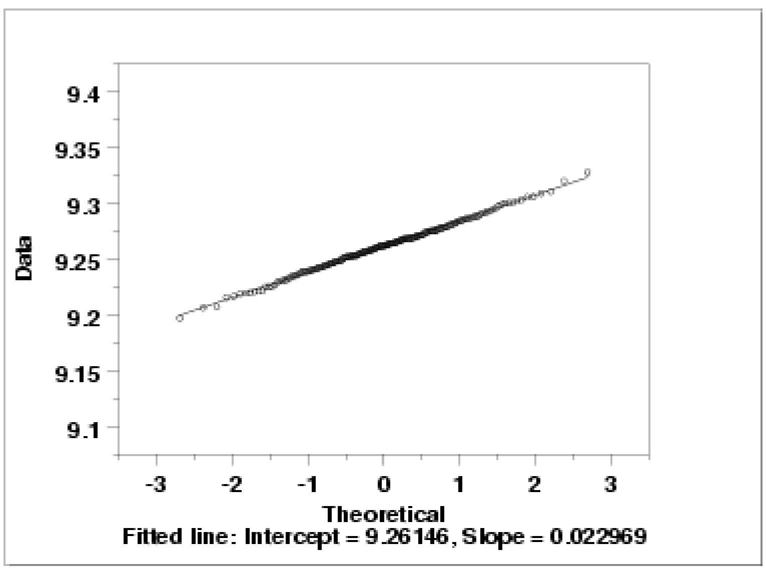
Lag Plot



Histogram (with overlaid Normal PDF)



Normal Probability Plot



1.4.2.8.3. Quantitative Output and Interpretation

Summary Statistics

As a first step in the analysis, common summary statistics are computed from the data.

```
Sample size  = 195
Mean         =   9.261460
Median       =   9.261952
Minimum      =   9.196848
Maximum      =   9.327973
Range        =   0.131126
Stan. Dev.   =   0.022789
```

Location

One way to quantify a change in location over time is to fit a straight line to the data using an index variable as the independent variable in the regression. For our data, we assume that data are in sequential run order and that the data were collected at equally spaced time intervals. In our regression, we use the index variable X = 1, 2, ..., N, where N is the number of observations. If there is no significant drift in the location over time, the slope parameter should be zero.

Coefficient	Estimate	Stan. Error	*t*-Value
B_0	9.26699	0.3253E-02	2849.
B_1	-0.56412E-04	0.2878E-04	-1.960

```
Residual Standard Deviation = 0.2262372E-01
Residual Degrees of Freedom = 193
```

The slope parameter, B_1, has a *t* value of -1.96 which is (barely) statistically significant since it is essentially equal to the 95 % level cutoff of -1.96. However, notice that the value of the slope parameter estimate is -0.00056. This slope, even though statistically significant, can essentially be considered zero.

Variation

One simple way to detect a change in variation is with a Bartlett test after dividing the data set into several equal-sized intervals. The choice of the number of intervals is somewhat arbitrary, although values of four or eight are reasonable. We will divide our data into four intervals.

2 2 2 2

H_0: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$
H_a: At least one σ_i^2 is not equal to the others.

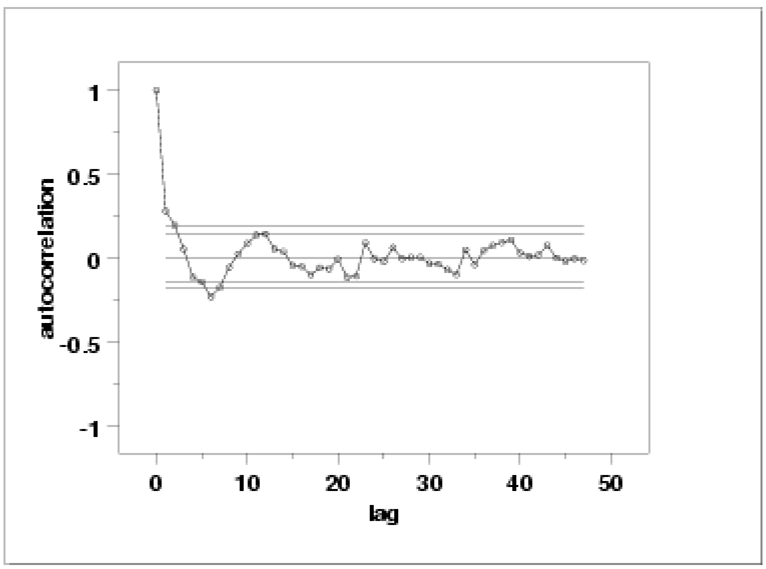
Test statistic: $T = 3.147$
Degrees of freedom: $k - 1 = 3$
Significance level: $\alpha = 0.05$
Critical value: $\chi^2_{1-\alpha,k-1} = 7.815$
Critical region: Reject H_0 if $T > 7.815$

In this case, since the Bartlett test statistic of 3.147 is less than the critical value at the 5 % significance level of 7.815, we conclude that the variances are not significantly different in the four intervals. That is, the assumption of constant scale is valid.

Randomness

There are many ways in which data can be non-random. However, most common forms of non-randomness can be detected with a few simple tests. The lag plot in the previous section is a simple graphical technique.

Another check is an autocorrelation plot that shows the autocorrelations for various lags. Confidence bands can be plotted at the 95 % and 99 % confidence levels. Points outside this band indicate statistically significant values (lag 0 is always 1).



The lag 1 autocorrelation, which is generally the one of greatest interest, is 0.281. The critical values at the 5 % significance level are -0.087 and 0.087. This indicates that the lag 1 autocorrelation is statistically significant, so there is evidence of non-randomness.

A common test for randomness is the runs test.

H_0: the sequence was produced in a random manner
H_a: the sequence was not produced in a random manner

Test statistic: $Z = -3.2306$
Significance level: $\alpha = 0.05$

```
Critical value:  Z_{1-α/2} = 1.96
Critical region:  Reject H_0 if |Z| > 1.96
```

The value of the test statistic is less than -1.96, so we reject the null hypothesis at the 0.05 significant level and conclude that the data are not random.

Although the autocorrelation plot and the runs test indicate some mild non-randomness, the violation of the randomness assumption is not serious enough to warrant developing a more sophisticated model. It is common in practice that some of the assumptions are mildly violated and it is a judgement call as to whether or not the violations are serious enough to warrant developing a more sophisticated model for the data.

Distributional Analysis

Probability plots are a graphical test for assessing if a particular distribution provides an adequate fit to a data set.

A quantitative enhancement to the probability plot is the correlation coefficient of the points on the probability plot. For this data set the correlation coefficient is 0.996. Since this is greater than the critical value of 0.987 (this is a tabulated value), the normality assumption is not rejected.

Chi-square and Kolmogorov-Smirnov goodness-of-fit tests are alternative methods for assessing distributional adequacy. The Wilk-Shapiro and Anderson-Darling tests can be used to test for normality. The results of the Anderson-Darling test follow.

```
H_0:  the data are normally distributed
H_a:  the data are not normally distributed

Adjusted test statistic:  A^2 = 0.129
Significance level:  α = 0.05
Critical value:  0.787
Critical region:  Reject H_0 if A^2 > 0.787
```

The Anderson-Darling test also does not reject the normality assumption because the test statistic, 0.129, is less than the critical value at the 5 % significance level of 0.787.

Outlier Analysis

A test for outliers is the Grubbs' test.

```
H_0:  there are no outliers in the data
H_a:  the maximum value is an outlier

Test statistic:  G = 2.918673
Significance level:  α = 0.05
Critical value for an upper one-tailed test:  3.597898
Critical region:  Reject H_0 if G > 3.597898
```

For this data set, Grubbs' test does not detect any outliers at the 0.05 significance level.

Model

Since the underlying assumptions were validated both graphically and analytically, with a mild violation of the randomness assumption, we conclude that a reasonable model for the data is:

$$Y_i = 9.26146 + E_i$$

We can express the uncertainty for ***C***, here estimated by 9.26146, as the 95 % confidence interval (9.258242,9.26479).

Univariate Report

It is sometimes useful and convenient to summarize the above results in a report. The report for the heat flow meter data follows.

```
 Analysis for heat flow meter data

 1: Sample Size                              = 195

 2: Location
    Mean                                      =
9.26146
    Standard Deviation of Mean               =
0.001632
    95 % Confidence Interval for Mean        =
(9.258242,9.264679)
    Drift with respect to location?          = NO

 3: Variation
    Standard Deviation                       =
0.022789
    95 % Confidence Interval for SD          =
(0.02073,0.025307)
    Drift with respect to variation?
    (based on Bartlett's test on quarters
    of the data)                              = NO

 4: Randomness
    Autocorrelation                          =
0.280579
    Data are Random?
      (as measured by autocorrelation)       = NO

 5: Data are Normal?
      (as tested by Anderson-Darling)        = YES

 6: Statistical Control
    (i.e., no drift in location or scale,
    data are random, distribution is
    fixed, here we are testing only for
    fixed normal)
    Data Set is in Statistical Control?      = YES

 7: Outliers?
    (as determined by Grubbs' test)           = NO
```

1.4.2.8.4. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to repeat the analysis outlined in the case study description on the previous page using Dataplot . It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	Results and Conclusions
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step.	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
`1. Invoke Dataplot and read data.` `1. Read in the data.`	`1. You have read 1 column of numbers into Dataplot, variable Y.`
`2. 4-plot of the data.` `1. 4-plot of Y.`	`1. Based on the 4-plot, there are no shifts in location or scale, and the data seem to follow a normal distribution.`
`3. Generate the individual plots.`	

1. Generate a run sequence plot. 2. Generate a lag plot. 3. Generate a histogram with an overlaid normal pdf. 4. Generate a normal probability plot.	1. The run sequence plot indicates that there are no shifts of location or scale. 2. The lag plot does not indicate any significant patterns (which would show the data were not random). 3. The histogram indicates that a normal distribution is a good distribution for these data. 4. The normal probability plot verifies that the normal distribution is a reasonable distribution for these data.
4. Generate summary statistics, quantitative analysis, and print a univariate report. 1. Generate a table of summary statistics. 2. Generate the mean, a confidence interval for the mean, and compute a linear fit to detect drift in location. 3. Generate the standard deviation, a confidence interval for the standard deviation, and detect drift in variation by dividing the data into quarters and computing Bartlett's test for equal standard deviations. 4. Check for randomness by generating an autocorrelation plot and a runs test. 5. Check for normality by computing the normal probability plot correlation coefficient. 6. Check for outliers using Grubbs' test. 7. Print a univariate report (this	1. The summary statistics table displays 25+ statistics. 2. The mean is 9.261 and a 95% confidence interval is (9.258,9.265). The linear fit indicates no drift in location since the slope parameter estimate is essentially zero. 3. The standard deviation is 0.023 with a 95% confidence interval of (0.0207,0.0253). Bartlett's test indicates no significant change in variation. 4. The lag 1 autocorrelation is 0.28. From the autocorrelation plot, this is statistically significant at the 95% level.

assumes steps 2 thru 6 have already been run).	5. The normal probability plot correlation coefficient is 0.999. At the 5% level, we cannot reject the normality assumption. 6. Grubbs' test detects no outliers at the 5% level. 7. The results are summarized in a convenient report.

1.4.2.9. Fatigue Life of Aluminum Alloy Specimens

Fatigue Life of Aluminum Alloy Specimens

This example illustrates the univariate analysis of the fatigue life of aluminum alloy specimens.

1.4.2.9.1. Background and Data

Generation

This data set comprises measurements of fatigue life (thousands of cycles until rupture) of rectangular strips of 6061-T6 aluminum sheeting, subjected to periodic loading with maximum stress of 21,000 psi (pounds per square inch), as reported by Birnbaum and Saunders (1958).

Purpose of Analysis

The goal of this case study is to select a probabilistic model, from among several reasonable alternatives, to describe the dispersion of the resulting measured values of life-length.

The original study, in the field of statistical reliability analysis, was concerned with the prediction of failure times of a material subjected to a load varying in time. It was well-known that a structure designed to withstand a particular static load may fail sooner than expected under a dynamic load.

If a realistic model for the probability distribution of lifetime can be found, then it can be used to estimate the time by which a part or structure needs to be replaced to guarantee that the probability of failure does not exceed some maximum acceptable value, for example 0.1 %, while it is in service.

The chapter of this eHandbook that is concerned with the assessment of product reliability contains additional material on statistical methods used in reliability analysis. This case study is meant to complement that chapter by showing the use of graphical and other techniques in the model selection stage of such analysis.

When there is no cogent reason to adopt a particular model, or when none of the models under consideration seems adequate for the purpose, one may opt for a non-parametric statistical method, for example to produce tolerance bounds or confidence intervals.

A non-parametric method does not rely on the assumption that the data are like a sample from a particular probability distribution that is fully specified up to the values of some adjustable parameters. For example, the Gaussian probability distribution is a parametric model with two adjustable parameters.

The price to be paid when using non-parametric methods is loss of efficiency, meaning that they may require more data for statistical inference than a parametric counterpart would, if applicable. For example, non-parametric confidence intervals for model parameters may be considerably wider than what a confidence interval would need to be if the underlying distribution could be identified correctly. Such identification is what we will attempt in this case study.

It should be noted --- a point that we will stress later in the development of this case study --- that the very exercise of selecting a model often contributes substantially to the uncertainty of the conclusions derived after the selection has been made.

Software

The analyses used in this case study can be generated using R code.

Data

The following data are used for this case study.

```
 370 1016 1235 1419 1567 1820
 706 1018 1238 1420 1578 1868
 716 1020 1252 1420 1594 1881
 746 1055 1258 1450 1602 1890
 785 1085 1262 1452 1604 1893
 797 1102 1269 1475 1608 1895
 844 1102 1270 1478 1630 1910
 855 1108 1290 1481 1642 1923
 858 1115 1293 1485 1674 1940
 886 1120 1300 1502 1730 1945
 886 1134 1310 1505 1750 2023
 930 1140 1313 1513 1750 2100
 960 1199 1315 1522 1763 2130
 988 1200 1330 1522 1768 2215
 990 1200 1355 1530 1781 2268
1000 1203 1390 1540 1782 2440
1010 1222 1416 1560 1792
```

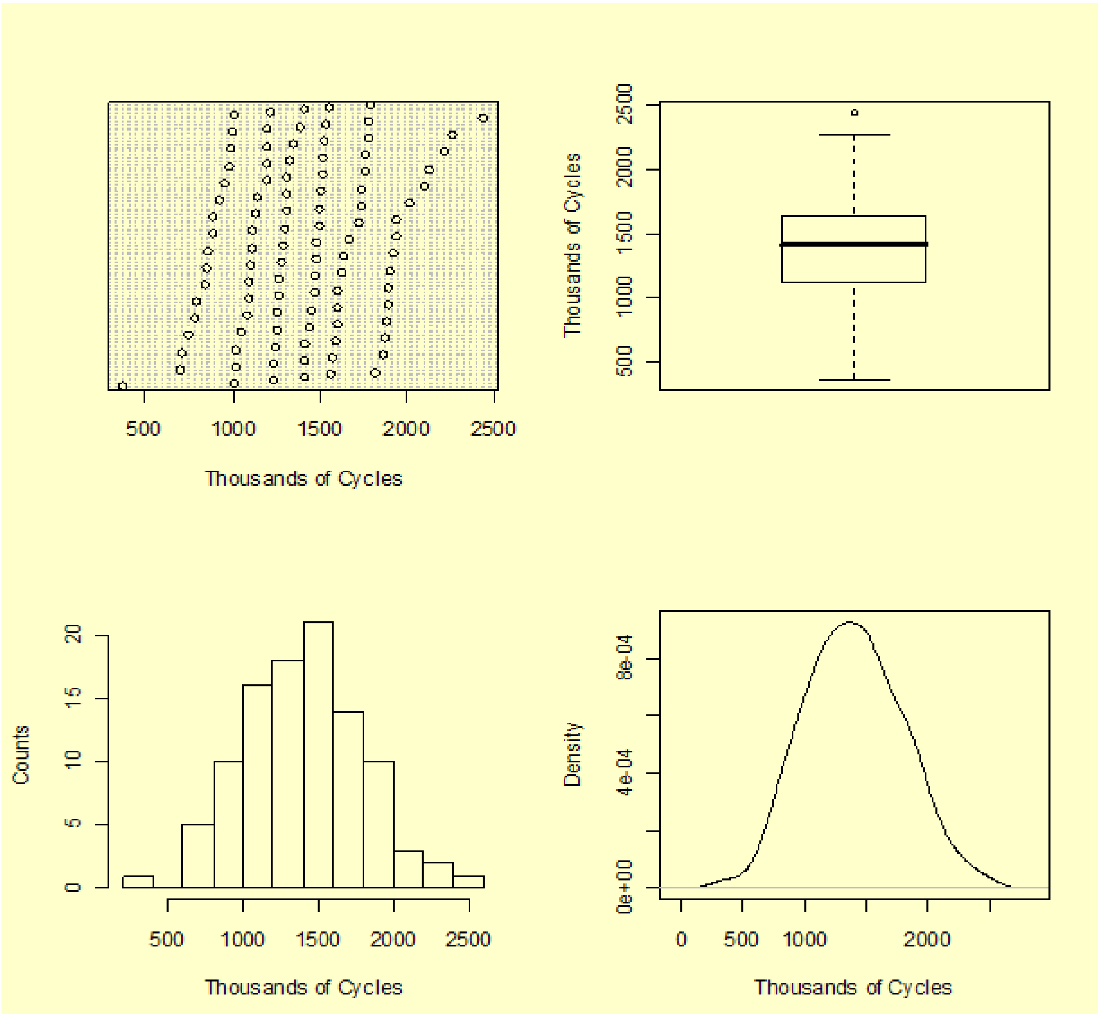
1.4.2.9.2. Graphical Output and Interpretation

Goal

The goal of this analysis is to select a probabilistic model to describe the dispersion of the measured values of fatigue life of specimens of an aluminum alloy described in [1.4.2.9.1], from among several reasonable alternatives.

Initial Plots of the Data

Simple diagrams can be very informative about location, spread, and to detect possibly anomalous data values or particular patterns (clustering, for example). These include dot-charts, boxplots, and histograms. Since building an effective histogram requires that a choice be made of bin size, and this choice can be influential, one may wish to examine a non-parametric estimate of the underlying probability density.

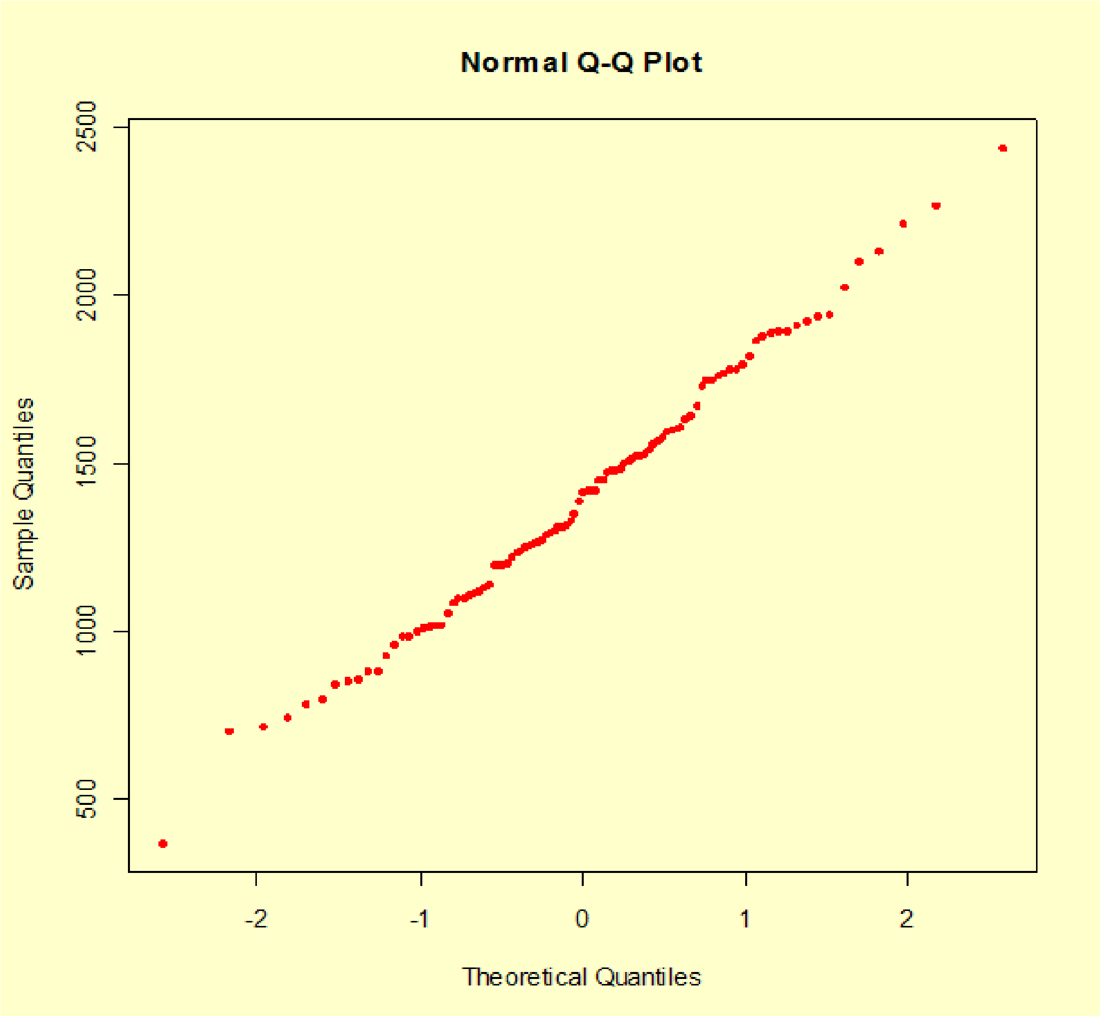


These several plots variously show that the measurements range from a value slightly greater than 350,000 to slightly less than 2,500,000 cycles. The boxplot suggests that the largest measured value may be an outlier.

A recommended first step is to check consistency between the data and what is to be expected if the data were a sample from a particular probability distribution. Knowledge about the underlying properties of materials and of relevant industrial processes typically offer clues as to the models that should be entertained. Graphical diagnostic techniques can be very useful at this exploratory stage: foremost among these, for univariate data, is the quantile-quantile plot, or QQ-plot (Wilk and Gnanadesikan, 1968).

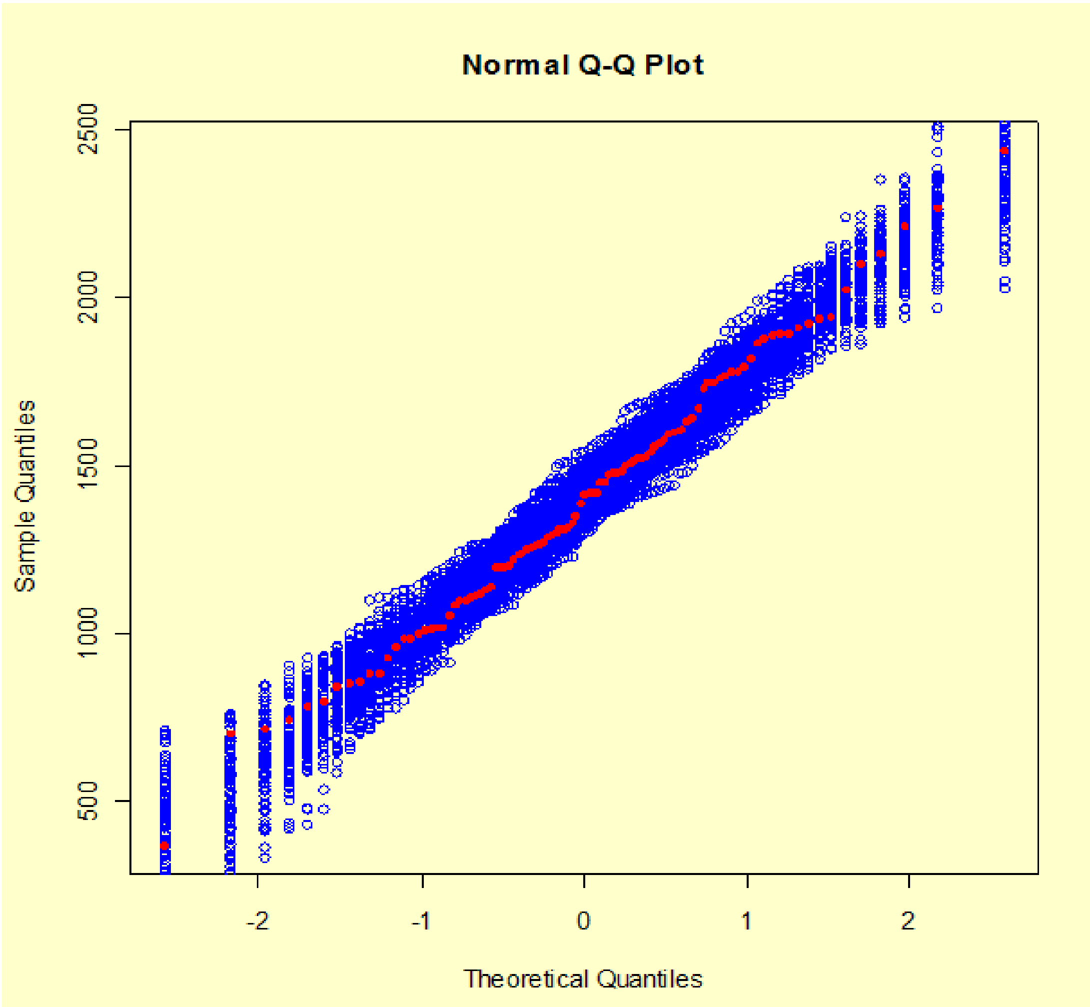
Each data point is represented by one point in the QQ-plot. The ordinate of each of these points is one data value; if this data value happens to be the *k*th order statistic in the sample (that is, the *k*th largest value), then the corresponding abscissa is the "typical" value that the *k*th largest value should have in a sample of the same size as the data, drawn from a particular distribution. If F denotes the cumulative probability distribution function of interest, and the sample comprises n values, then $F^{-1}[(k - 1/2) / (n + 1/2)]$ is a reasonable choice for that "typical" value, because it is an approximation to the median of the *k*th order statistic in a sample of size n from this distribution.

The following figure shows a QQ-plot of our data relative to the Gaussian (or, normal) probability distribution. If the data matched expectations perfectly, then the points would all fall on a straight line.



In practice, one needs to gauge whether the deviations from such perfect alignment are commensurate with the natural variability associated with sampling. This can easily be done by examining how variable QQ-plots of samples from the target distribution may be.

The following figure shows, superimposed on the QQ-plot of the data, the QQ-plots of 99 samples of the same size as the data, drawn from a Gaussian distribution with the same mean and standard deviation as the data.



The fact that the cloud of QQ-plots corresponding to 99 samples from the Gaussian distribution effectively covers the QQ-plot for the data, suggests that the chances are better than 1 in 100 that our data are inconsistent with the Gaussian model.

This proves nothing, of course, because even the rarest of events may happen. However, it is commonly taken to be indicative of an acceptable fit for general purposes. In any case, one may naturally wonder if an alternative model might not provide an even better fit.

Knowing the provenance of the data, that they portray strength of a material, strongly suggests that one may like to examine alternative models, because in many studies of reliability non-Gaussian models tend to be more appropriate than Gaussian models.

Candidate Distributions

There are many probability distributions that could reasonably be entertained as candidate models for the data. However, we will restrict ourselves to consideration of the following because these have proven to be useful in reliability studies.

- Normal distribution
- Gamma distribution
- Birnbaum-Saunders distribution
- 3-parameter Weibull distribution

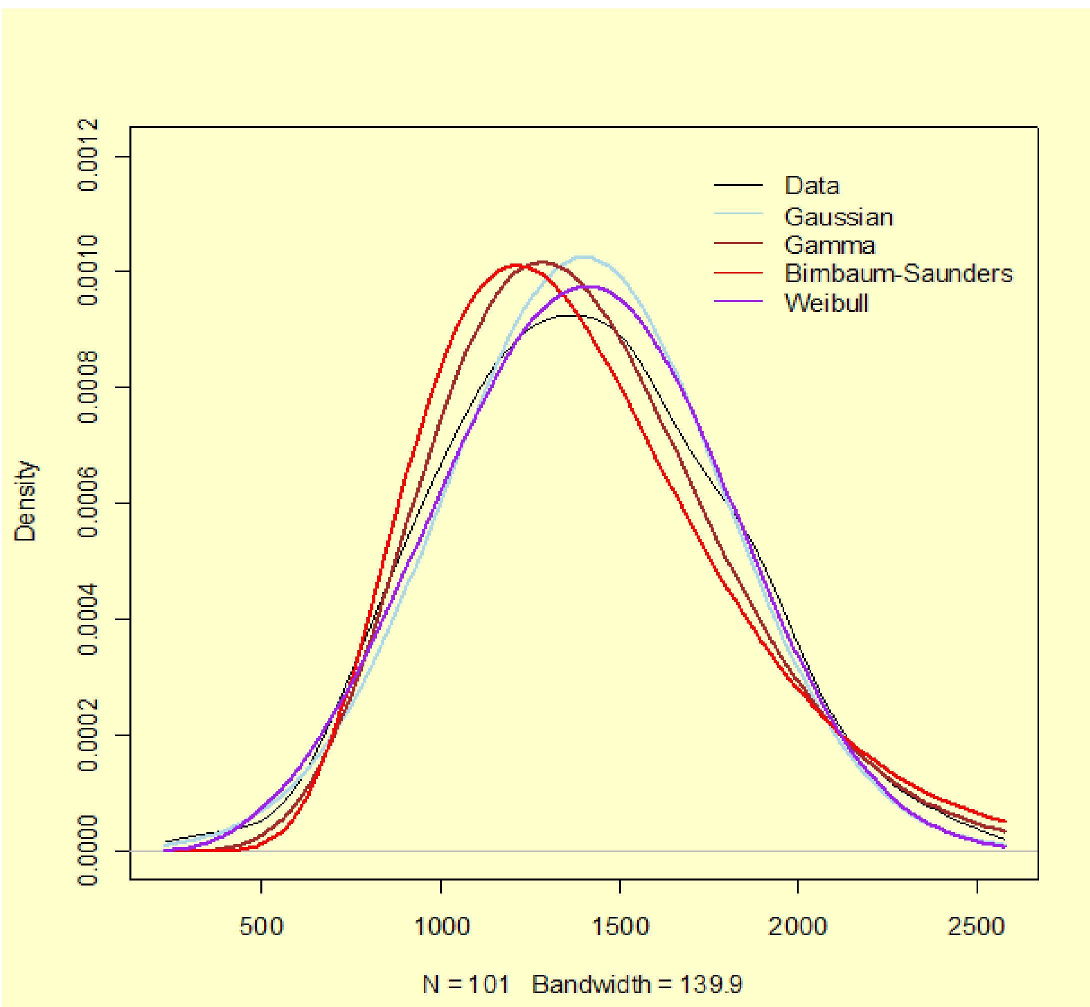
Approach

A very simple approach amounts to comparing QQ-plots of the data for the candidate models under consideration. This typically involves first fitting the models to the data, for example employing the method of maximum likelihood [1.3.6.5.2].

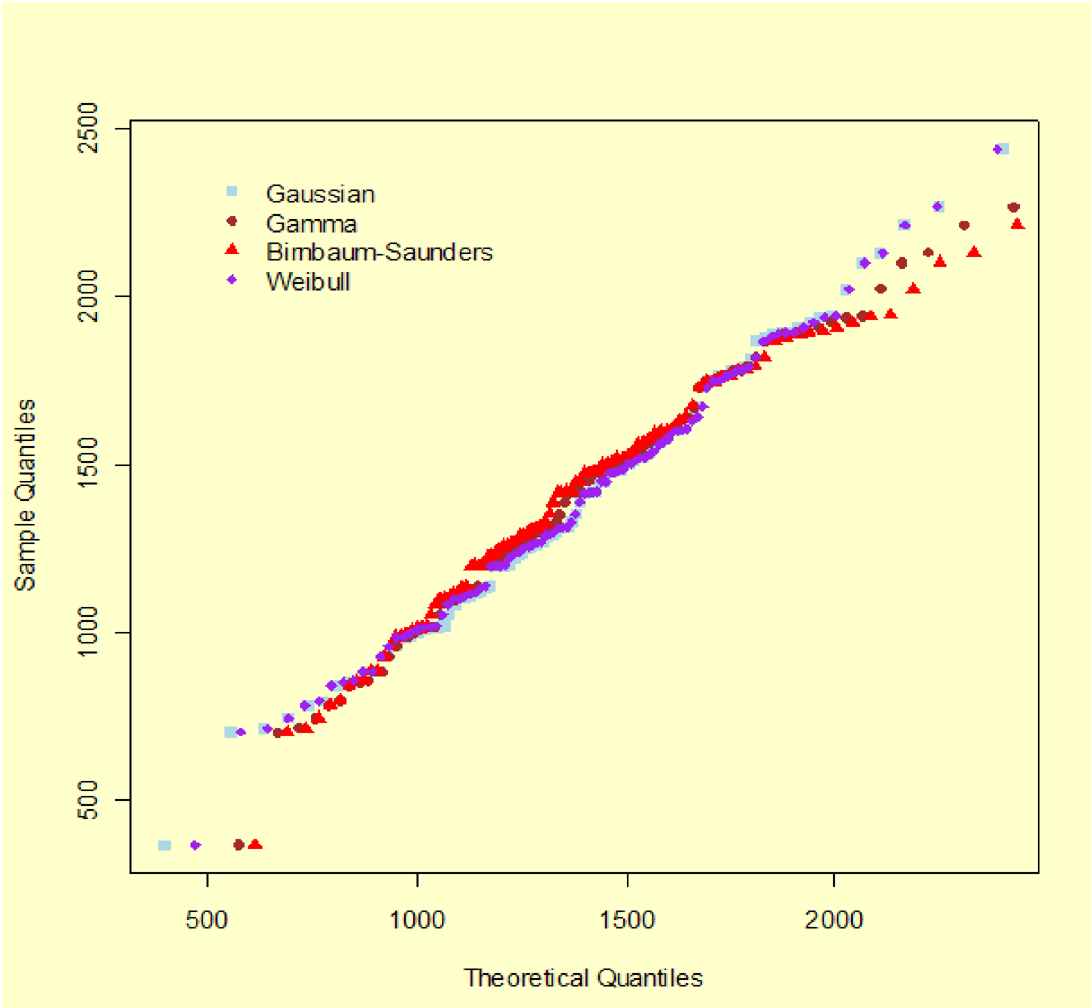
The maximum likelihood estimates are the following:

- Gaussian: mean 1401, standard deviation 389
- Gamma: shape 11.85, rate 0.00846
- Birnbaum-Saunders: shape 0.310, scale 1337
- 3-parameter Weibull: location 181, shape 3.43, scale 1357

The following figure shows how close (or how far) the best fitting probability densities of the four distributions approximate the non-parametric probability density estimate. This comparison, however, takes into account neither the fact that our sample is fairly small (101 measured values), nor that the fitted models themselves have been estimated from the same data that the non-parametric estimate was derived from.



These limitations notwithstanding, it is worth examining the corresponding QQ-plots, shown below, which suggest that the Gaussian and the 3-parameter Weibull may be the best models.



Model Selection

A more careful comparison of the merits of the alternative models needs to take into account the fact that the 3-parameter Weibull model (precisely because it has three parameters), may be intrinsically more flexible than the others, which all have two adjustable parameters only.

Two criteria can be employed for a formal comparison: Akaike's Information Criterion (AIC), and the Bayesian Information Criterion (BIC) (Hastie et. al., 2001). The smaller the value of either model selection criterion, the better the model:

```
     AIC  BIC
GAU 1495 1501
GAM 1499 1504
BS  1507 1512
WEI 1498 1505
```

On this basis (and according both to AIC and BIC), there seems to be no cogent reason to replace the Gaussian model by any of the other three. The values of BIC can also be used to derive an approximate answer to the question of how strongly the data may support each of these models. Doing this involves the application of Bayesian statistical methods [8.1.10].

We start from an *a priori* assignment of equal probabilities to all four models, indicating that we have no reason to favor one over another at the outset, and then update these probabilities based on the measured values of lifetime. The updated probabilities of the four models, called their *posterior probabilities*, are approximately proportional to exp(-BIC(GAU)/2), exp(-BIC(GAM)/2), exp(-BIC(BS)/2), and exp(-BIC(WEI)/2). The values are 76 % for GAU, 16 % for GAM, 0.27 % for BS, and 7.4 % for WEI.

One possible use for the selected model is to answer the question of the age in service by which a part or structure needs to be replaced to guarantee that the probability of failure does not exceed some maximum acceptable value, for example 0.1 %.The answer to this question is the 0.1st percentile of the fitted distribution, that is $G^{-1}(0.001)$ = 198 thousand cycles, where, in this case, G^{-1} denotes the inverse of the fitted, Gaussian probability distribution.

To assess the uncertainty of this estimate one may employ the statistical bootstrap [1.3.3.4]. In this case, this involves drawing a suitably large number of bootstrap samples from the data, and for each of them applying the model fitting and model selection exercise described above, ending with the calculation of $G^{-1}(0.001)$ for the best model (which may vary from sample to sample).

The bootstrap samples should be of the same size as the data, with each being drawn uniformly at random from the data, *with* replacement. This process, based on 5,000 bootstrap samples, yielded a 95 % confidence interval for the 0.1st percentile ranging from 40 to 366 thousands of cycles. The large uncertainty is not surprising given that we are attempting to estimate the largest value that is exceeded with probability 99.9 %, based on a sample comprising only 101 measured values.

Prediction Intervals

One more application in this analysis is to evaluate prediction intervals for the fatigue life of the aluminum alloy specimens. For example, if we were to test three new specimens using the same process, we would want to know (with 95 % confidence) the minimum number of cycles for these three specimens. That is, we need to find a statistical interval $[L, \infty]$ that contains the fatigue life of all three future specimens with 95 % confidence. The desired interval is a one-sided, lower 95 % prediction interval. Since tables of factors for constructing L, are widely available for normal models, we use the results corresponding to the normal model here for illustration. Specifically, L is computed as

$$\begin{aligned} L &= \bar{x} + rs \\ L &= 1400.91 - 2.16(391.32) = 555.66 \text{ cycles} \times 1000 \end{aligned}$$

where factor r is given in Table A.14 of Hahn and Meeker (1991) or can be obtained from an R program.

1.4.2.10. Ceramic Strength

Ceramic Strength

This case study analyzes the effect of machining factors on the strength of ceramics.

1.4.2.10.1. Background and Data

Generation

The data for this case study were collected by Said Jahanmir of the NIST Ceramics Division in 1996 in connection with a NIST/industry ceramics consortium for strength optimization of ceramic strength

The motivation for studying this data set is to illustrate the analysis of multiple factors from a designed experiment

This case study will utilize only a subset of a full study that was conducted by Lisa Gill and James Filliben of the NIST Statistical Engineering Division

The response variable is a measure of the strength of the ceramic material (bonded S_i nitrate). The complete data set contains the following variables:

1. Factor 1 = Observation ID, i.e., run number (1 to 960)
2. Factor 2 = Lab (1 to 8)
3. Factor 3 = Bar ID within lab (1 to 30)
4. Factor 4 = Test number (1 to 4)
5. Response Variable = Strength of Ceramic
6. Factor 5 = Table speed (2 levels: 0.025 and 0.125)
7. Factor 6 = Down feed rate (2 levels: 0.050 and 0.125)
8. Factor 7 = Wheel grit size (2 levels: 150 and 80)
9. Factor 8 = Direction (2 levels: longitudinal and transverse)
10. Factor 9 = Treatment (1 to 16)
11. Factor 10 = Set of 15 within lab (2 levels: 1 and 2)
12. Factor 11 = Replication (2 levels: 1 and 2)
13. Factor 12 = Bar Batch (1 and 2)

The four primary factors of interest are:

1. Table speed (X1)
2. Down feed rate (X2)
3. Wheel grit size (X3)
4. Direction (X4)

For this case study, we are using only half the data. Specifically, we are using the data with the direction longitudinal. Therefore, we have only three primary factors

In addition, we are interested in the nuisance factors

1. Lab
2. Batch

Purpose of Analysis

The goals of this case study are:

1. Determine which of the four primary factors has the strongest effect on the strength of the ceramic material
2. Estimate the magnitude of the effects
3. Determine the optimal settings for the primary factors
4. Determine if the nuisance factors (lab and batch) have an effect on the ceramic strength

This case study is an example of a designed experiment. The Process Improvement chapter contains a detailed discussion of the construction and analysis of designed experiments. This case study is meant to complement the material in that chapter by showing how an EDA approach (emphasizing the use of graphical techniques) can be used in the analysis of designed experiments

Software

The analyses used in this case study can be generated using both Dataplot code and R code.

Data

The following are the data used for this case study

Run	Lab	Batch	Y	X1	X2	X3
1	1	1	608.781	-1	-1	-1
2	1	2	569.670	-1	-1	-1
3	1	1	689.556	-1	-1	-1
4	1	2	747.541	-1	-1	-1
5	1	1	618.134	-1	-1	-1
6	1	2	612.182	-1	-1	-1
7	1	1	680.203	-1	-1	-1
8	1	2	607.766	-1	-1	-1
9	1	1	726.232	-1	-1	-1
10	1	2	605.380	-1	-1	-1
11	1	1	518.655	-1	-1	-1
12	1	2	589.226	-1	-1	-1
13	1	1	740.447	-1	-1	-1
14	1	2	588.375	-1	-1	-1
15	1	1	666.830	-1	-1	-1
16	1	2	531.384	-1	-1	-1
17	1	1	710.272	-1	-1	-1
18	1	2	633.417	-1	-1	-1
19	1	1	751.669	-1	-1	-1
20	1	2	619.060	-1	-1	-1
21	1	1	697.979	-1	-1	-1
22	1	2	632.447	-1	-1	-1
23	1	1	708.583	-1	-1	-1
24	1	2	624.256	-1	-1	-1
25	1	1	624.972	-1	-1	-1
26	1	2	575.143	-1	-1	-1
27	1	1	695.070	-1	-1	-1
28	1	2	549.278	-1	-1	-1
29	1	1	769.391	-1	-1	-1
30	1	2	624.972	-1	-1	-1
61	1	1	720.186	-1	1	1
62	1	2	587.695	-1	1	1
63	1	1	723.657	-1	1	1
64	1	2	569.207	-1	1	1
65	1	1	703.700	-1	1	1
66	1	2	613.257	-1	1	1
67	1	1	697.626	-1	1	1
68	1	2	565.737	-1	1	1

69	1	1	714.980	-1	1	1
70	1	2	662.131	-1	1	1
71	1	1	657.712	-1	1	1
72	1	2	543.177	-1	1	1
73	1	1	609.989	-1	1	1
74	1	2	512.394	-1	1	1
75	1	1	650.771	-1	1	1
76	1	2	611.190	-1	1	1
77	1	1	707.977	-1	1	1
78	1	2	659.982	-1	1	1
79	1	1	712.199	-1	1	1
80	1	2	569.245	-1	1	1
81	1	1	709.631	-1	1	1
82	1	2	725.792	-1	1	1
83	1	1	703.160	-1	1	1
84	1	2	608.960	-1	1	1
85	1	1	744.822	-1	1	1
86	1	2	586.060	-1	1	1
87	1	1	719.217	-1	1	1
88	1	2	617.441	-1	1	1
89	1	1	619.137	-1	1	1
90	1	2	592.845	-1	1	1
151	2	1	753.333	1	1	1
152	2	2	631.754	1	1	1
153	2	1	677.933	1	1	1
154	2	2	588.113	1	1	1
155	2	1	735.919	1	1	1
156	2	2	555.724	1	1	1
157	2	1	695.274	1	1	1
158	2	2	702.411	1	1	1
159	2	1	504.167	1	1	1
160	2	2	631.754	1	1	1
161	2	1	693.333	1	1	1
162	2	2	698.254	1	1	1
163	2	1	625.000	1	1	1
164	2	2	616.791	1	1	1
165	2	1	596.667	1	1	1
166	2	2	551.953	1	1	1
167	2	1	640.898	1	1	1
168	2	2	636.738	1	1	1
169	2	1	720.506	1	1	1
170	2	2	571.551	1	1	1
171	2	1	700.748	1	1	1
172	2	2	521.667	1	1	1
173	2	1	691.604	1	1	1
174	2	2	587.451	1	1	1
175	2	1	636.738	1	1	1
176	2	2	700.422	1	1	1
177	2	1	731.667	1	1	1
178	2	2	595.819	1	1	1
179	2	1	635.079	1	1	1
180	2	2	534.236	1	1	1
181	2	1	716.926	1	-1	-1
182	2	2	606.188	1	-1	-1
183	2	1	759.581	1	-1	-1
184	2	2	575.303	1	-1	-1
185	2	1	673.903	1	-1	-1
186	2	2	590.628	1	-1	-1
187	2	1	736.648	1	-1	-1
188	2	2	729.314	1	-1	-1
189	2	1	675.957	1	-1	-1
190	2	2	619.313	1	-1	-1
191	2	1	729.230	1	-1	-1
192	2	2	624.234	1	-1	-1
193	2	1	697.239	1	-1	-1
194	2	2	651.304	1	-1	-1
195	2	1	728.499	1	-1	-1
196	2	2	724.175	1	-1	-1
197	2	1	797.662	1	-1	-1
198	2	2	583.034	1	-1	-1
199	2	1	668.530	1	-1	-1
200	2	2	620.227	1	-1	-1
201	2	1	815.754	1	-1	-1
202	2	2	584.861	1	-1	-1
203	2	1	777.392	1	-1	-1
204	2	2	565.391	1	-1	-1
205	2	1	712.140	1	-1	-1
206	2	2	622.506	1	-1	-1
207	2	1	663.622	1	-1	-1
208	2	2	628.336	1	-1	-1
209	2	1	684.181	1	-1	-1

210	2	2	587.145	1	-1	-1
271	3	1	629.012	1	-1	1
272	3	2	584.319	1	-1	1
273	3	1	640.193	1	-1	1
274	3	2	538.239	1	-1	1
275	3	1	644.156	1	-1	1
276	3	2	538.097	1	-1	1
277	3	1	642.469	1	-1	1
278	3	2	595.686	1	-1	1
279	3	1	639.090	1	-1	1
280	3	2	648.935	1	-1	1
281	3	1	439.418	1	-1	1
282	3	2	583.827	1	-1	1
283	3	1	614.664	1	-1	1
284	3	2	534.905	1	-1	1
285	3	1	537.161	1	-1	1
286	3	2	569.858	1	-1	1
287	3	1	656.773	1	-1	1
288	3	2	617.246	1	-1	1
289	3	1	659.534	1	-1	1
290	3	2	610.337	1	-1	1
291	3	1	695.278	1	-1	1
292	3	2	584.192	1	-1	1
293	3	1	734.040	1	-1	1
294	3	2	598.853	1	-1	1
295	3	1	687.665	1	-1	1
296	3	2	554.774	1	-1	1
297	3	1	710.858	1	-1	1
298	3	2	605.694	1	-1	1
299	3	1	701.716	1	-1	1
300	3	2	627.516	1	-1	1
301	3	1	382.133	1	1	-1
302	3	2	574.522	1	1	-1
303	3	1	719.744	1	1	-1
304	3	2	582.682	1	1	-1
305	3	1	756.820	1	1	-1
306	3	2	563.872	1	1	-1
307	3	1	690.978	1	1	-1
308	3	2	715.962	1	1	-1
309	3	1	670.864	1	1	-1
310	3	2	616.430	1	1	-1
311	3	1	670.308	1	1	-1
312	3	2	778.011	1	1	-1
313	3	1	660.062	1	1	-1
314	3	2	604.255	1	1	-1
315	3	1	790.382	1	1	-1
316	3	2	571.906	1	1	-1
317	3	1	714.750	1	1	-1
318	3	2	625.925	1	1	-1
319	3	1	716.959	1	1	-1
320	3	2	682.426	1	1	-1
321	3	1	603.363	1	1	-1
322	3	2	707.604	1	1	-1
323	3	1	713.796	1	1	-1
324	3	2	617.400	1	1	-1
325	3	1	444.963	1	1	-1
326	3	2	689.576	1	1	-1
327	3	1	723.276	1	1	-1
328	3	2	676.678	1	1	-1
329	3	1	745.527	1	1	-1
330	3	2	563.290	1	1	-1
361	4	1	778.333	-1	-1	1
362	4	2	581.879	-1	-1	1
363	4	1	723.349	-1	-1	1
364	4	2	447.701	-1	-1	1
365	4	1	708.229	-1	-1	1
366	4	2	557.772	-1	-1	1
367	4	1	681.667	-1	-1	1
368	4	2	593.537	-1	-1	1
369	4	1	566.085	-1	-1	1
370	4	2	632.585	-1	-1	1
371	4	1	687.448	-1	-1	1
372	4	2	671.350	-1	-1	1
373	4	1	597.500	-1	-1	1
374	4	2	569.530	-1	-1	1
375	4	1	637.410	-1	-1	1
376	4	2	581.667	-1	-1	1
377	4	1	755.864	-1	-1	1
378	4	2	643.449	-1	-1	1
379	4	1	692.945	-1	-1	1
380	4	2	581.593	-1	-1	1

381	4	1	766.532	-1	-1	1
382	4	2	494.122	-1	-1	1
383	4	1	725.663	-1	-1	1
384	4	2	620.948	-1	-1	1
385	4	1	698.818	-1	-1	1
386	4	2	615.903	-1	-1	1
387	4	1	760.000	-1	-1	1
388	4	2	606.667	-1	-1	1
389	4	1	775.272	-1	-1	1
390	4	2	579.167	-1	-1	1
421	4	1	708.885	-1	1	-1
422	4	2	662.510	-1	1	-1
423	4	1	727.201	-1	1	-1
424	4	2	436.237	-1	1	-1
425	4	1	642.560	-1	1	-1
426	4	2	644.223	-1	1	-1
427	4	1	690.773	-1	1	-1
428	4	2	586.035	-1	1	-1
429	4	1	688.333	-1	1	-1
430	4	2	620.833	-1	1	-1
431	4	1	743.973	-1	1	-1
432	4	2	652.535	-1	1	-1
433	4	1	682.461	-1	1	-1
434	4	2	593.516	-1	1	-1
435	4	1	761.430	-1	1	-1
436	4	2	587.451	-1	1	-1
437	4	1	691.542	-1	1	-1
438	4	2	570.964	-1	1	-1
439	4	1	643.392	-1	1	-1
440	4	2	645.192	-1	1	-1
441	4	1	697.075	-1	1	-1
442	4	2	540.079	-1	1	-1
443	4	1	708.229	-1	1	-1
444	4	2	707.117	-1	1	-1
445	4	1	746.467	-1	1	-1
446	4	2	621.779	-1	1	-1
447	4	1	744.819	-1	1	-1
448	4	2	585.777	-1	1	-1
449	4	1	655.029	-1	1	-1
450	4	2	703.980	-1	1	-1
541	5	1	715.224	-1	-1	-1
542	5	2	698.237	-1	-1	-1
543	5	1	614.417	-1	-1	-1
544	5	2	757.120	-1	-1	-1
545	5	1	761.363	-1	-1	-1
546	5	2	621.751	-1	-1	-1
547	5	1	716.106	-1	-1	-1
548	5	2	472.125	-1	-1	-1
549	5	1	659.502	-1	-1	-1
550	5	2	612.700	-1	-1	-1
551	5	1	730.781	-1	-1	-1
552	5	2	583.170	-1	-1	-1
553	5	1	546.928	-1	-1	-1
554	5	2	599.771	-1	-1	-1
555	5	1	734.203	-1	-1	-1
556	5	2	549.227	-1	-1	-1
557	5	1	682.051	-1	-1	-1
558	5	2	605.453	-1	-1	-1
559	5	1	701.341	-1	-1	-1
560	5	2	569.599	-1	-1	-1
561	5	1	759.729	-1	-1	-1
562	5	2	637.233	-1	-1	-1
563	5	1	689.942	-1	-1	-1
564	5	2	621.774	-1	-1	-1
565	5	1	769.424	-1	-1	-1
566	5	2	558.041	-1	-1	-1
567	5	1	715.286	-1	-1	-1
568	5	2	583.170	-1	-1	-1
569	5	1	776.197	-1	-1	-1
570	5	2	345.294	-1	-1	-1
571	5	1	547.099	1	-1	1
572	5	2	570.999	1	-1	1
573	5	1	619.942	1	-1	1
574	5	2	603.232	1	-1	1
575	5	1	696.046	1	-1	1
576	5	2	595.335	1	-1	1
577	5	1	573.109	1	-1	1
578	5	2	581.047	1	-1	1
579	5	1	638.794	1	-1	1
580	5	2	455.878	1	-1	1
581	5	1	708.193	1	-1	1

582	5	2	627.880	1	-1	1
583	5	1	502.825	1	-1	1
584	5	2	464.085	1	-1	1
585	5	1	632.633	1	-1	1
586	5	2	596.129	1	-1	1
587	5	1	683.382	1	-1	1
588	5	2	640.371	1	-1	1
589	5	1	684.812	1	-1	1
590	5	2	621.471	1	-1	1
591	5	1	738.161	1	-1	1
592	5	2	612.727	1	-1	1
593	5	1	671.492	1	-1	1
594	5	2	606.460	1	-1	1
595	5	1	709.771	1	-1	1
596	5	2	571.760	1	-1	1
597	5	1	685.199	1	-1	1
598	5	2	599.304	1	-1	1
599	5	1	624.973	1	-1	1
600	5	2	579.459	1	-1	1
601	6	1	757.363	1	1	1
602	6	2	761.511	1	1	1
603	6	1	633.417	1	1	1
604	6	2	566.969	1	1	1
605	6	1	658.754	1	1	1
606	6	2	654.397	1	1	1
607	6	1	664.666	1	1	1
608	6	2	611.719	1	1	1
609	6	1	663.009	1	1	1
610	6	2	577.409	1	1	1
611	6	1	773.226	1	1	1
612	6	2	576.731	1	1	1
613	6	1	708.261	1	1	1
614	6	2	617.441	1	1	1
615	6	1	739.086	1	1	1
616	6	2	577.409	1	1	1
617	6	1	667.786	1	1	1
618	6	2	548.957	1	1	1
619	6	1	674.481	1	1	1
620	6	2	623.315	1	1	1
621	6	1	695.688	1	1	1
622	6	2	621.761	1	1	1
623	6	1	588.288	1	1	1
624	6	2	553.978	1	1	1
625	6	1	545.610	1	1	1
626	6	2	657.157	1	1	1
627	6	1	752.305	1	1	1
628	6	2	610.882	1	1	1
629	6	1	684.523	1	1	1
630	6	2	552.304	1	1	1
631	6	1	717.159	-1	1	-1
632	6	2	545.303	-1	1	-1
633	6	1	721.343	-1	1	-1
634	6	2	651.934	-1	1	-1
635	6	1	750.623	-1	1	-1
636	6	2	635.240	-1	1	-1
637	6	1	776.488	-1	1	-1
638	6	2	641.083	-1	1	-1
639	6	1	750.623	-1	1	-1
640	6	2	645.321	-1	1	-1
641	6	1	600.840	-1	1	-1
642	6	2	566.127	-1	1	-1
643	6	1	686.196	-1	1	-1
644	6	2	647.844	-1	1	-1
645	6	1	687.870	-1	1	-1
646	6	2	554.815	-1	1	-1
647	6	1	725.527	-1	1	-1
648	6	2	620.087	-1	1	-1
649	6	1	658.796	-1	1	-1
650	6	2	711.301	-1	1	-1
651	6	1	690.380	-1	1	-1
652	6	2	644.355	-1	1	-1
653	6	1	737.144	-1	1	-1
654	6	2	713.812	-1	1	-1
655	6	1	663.851	-1	1	-1
656	6	2	696.707	-1	1	-1
657	6	1	766.630	-1	1	-1
658	6	2	589.453	-1	1	-1
659	6	1	625.922	-1	1	-1
660	6	2	634.468	-1	1	-1
721	7	1	694.430	1	1	-1
722	7	2	599.751	1	1	-1

723	7	1	730.217	1	1	-1
724	7	2	624.542	1	1	-1
725	7	1	700.770	1	1	-1
726	7	2	723.505	1	1	-1
727	7	1	722.242	1	1	-1
728	7	2	674.717	1	1	-1
729	7	1	763.828	1	1	-1
730	7	2	608.539	1	1	-1
731	7	1	695.668	1	1	-1
732	7	2	612.135	1	1	-1
733	7	1	688.887	1	1	-1
734	7	2	591.935	1	1	-1
735	7	1	531.021	1	1	-1
736	7	2	676.656	1	1	-1
737	7	1	698.915	1	1	-1
738	7	2	647.323	1	1	-1
739	7	1	735.905	1	1	-1
740	7	2	811.970	1	1	-1
741	7	1	732.039	1	1	-1
742	7	2	603.883	1	1	-1
743	7	1	751.832	1	1	-1
744	7	2	608.643	1	1	-1
745	7	1	618.663	1	1	-1
746	7	2	630.778	1	1	-1
747	7	1	744.845	1	1	-1
748	7	2	623.063	1	1	-1
749	7	1	690.826	1	1	-1
750	7	2	472.463	1	1	-1
811	7	1	666.893	-1	1	1
812	7	2	645.932	-1	1	1
813	7	1	759.860	-1	1	1
814	7	2	577.176	-1	1	1
815	7	1	683.752	-1	1	1
816	7	2	567.530	-1	1	1
817	7	1	729.591	-1	1	1
818	7	2	821.654	-1	1	1
819	7	1	730.706	-1	1	1
820	7	2	684.490	-1	1	1
821	7	1	763.124	-1	1	1
822	7	2	600.427	-1	1	1
823	7	1	724.193	-1	1	1
824	7	2	686.023	-1	1	1
825	7	1	630.352	-1	1	1
826	7	2	628.109	-1	1	1
827	7	1	750.338	-1	1	1
828	7	2	605.214	-1	1	1
829	7	1	752.417	-1	1	1
830	7	2	640.260	-1	1	1
831	7	1	707.899	-1	1	1
832	7	2	700.767	-1	1	1
833	7	1	715.582	-1	1	1
834	7	2	665.924	-1	1	1
835	7	1	728.746	-1	1	1
836	7	2	555.926	-1	1	1
837	7	1	591.193	-1	1	1
838	7	2	543.299	-1	1	1
839	7	1	592.252	-1	1	1
840	7	2	511.030	-1	1	1
901	8	1	740.833	-1	-1	1
902	8	2	583.994	-1	-1	1
903	8	1	786.367	-1	-1	1
904	8	2	611.048	-1	-1	1
905	8	1	712.386	-1	-1	1
906	8	2	623.338	-1	-1	1
907	8	1	738.333	-1	-1	1
908	8	2	679.585	-1	-1	1
909	8	1	741.480	-1	-1	1
910	8	2	665.004	-1	-1	1
911	8	1	729.167	-1	-1	1
912	8	2	655.860	-1	-1	1
913	8	1	795.833	-1	-1	1
914	8	2	715.711	-1	-1	1
915	8	1	723.502	-1	-1	1
916	8	2	611.999	-1	-1	1
917	8	1	718.333	-1	-1	1
918	8	2	577.722	-1	-1	1
919	8	1	768.080	-1	-1	1
920	8	2	615.129	-1	-1	1
921	8	1	747.500	-1	-1	1
922	8	2	540.316	-1	-1	1
923	8	1	775.000	-1	-1	1

924	8	2	711.667	-1	-1	1
925	8	1	760.599	-1	-1	1
926	8	2	639.167	-1	-1	1
927	8	1	758.333	-1	-1	1
928	8	2	549.491	-1	-1	1
929	8	1	682.500	-1	-1	1
930	8	2	684.167	-1	-1	1
931	8	1	658.116	1	-1	-1
932	8	2	672.153	1	-1	-1
933	8	1	738.213	1	-1	-1
934	8	2	594.534	1	-1	-1
935	8	1	681.236	1	-1	-1
936	8	2	627.650	1	-1	-1
937	8	1	704.904	1	-1	-1
938	8	2	551.870	1	-1	-1
939	8	1	693.623	1	-1	-1
940	8	2	594.534	1	-1	-1
941	8	1	624.993	1	-1	-1
942	8	2	602.660	1	-1	-1
943	8	1	700.228	1	-1	-1
944	8	2	585.450	1	-1	-1
945	8	1	611.874	1	-1	-1
946	8	2	555.724	1	-1	-1
947	8	1	579.167	1	-1	-1
948	8	2	574.934	1	-1	-1
949	8	1	720.872	1	-1	-1
950	8	2	584.625	1	-1	-1
951	8	1	690.320	1	-1	-1
952	8	2	555.724	1	-1	-1
953	8	1	677.933	1	-1	-1
954	8	2	611.874	1	-1	-1
955	8	1	674.600	1	-1	-1
956	8	2	698.254	1	-1	-1
957	8	1	611.999	1	-1	-1
958	8	2	748.130	1	-1	-1
959	8	1	530.680	1	-1	-1
960	8	2	689.942	1	-1	-1

1. Exploratory Data Analysis
1.4. EDA Case Studies
1.4.2. Case Studies
1.4.2.10. Ceramic Strength

1.4.2.10.2. Analysis of the Response Variable

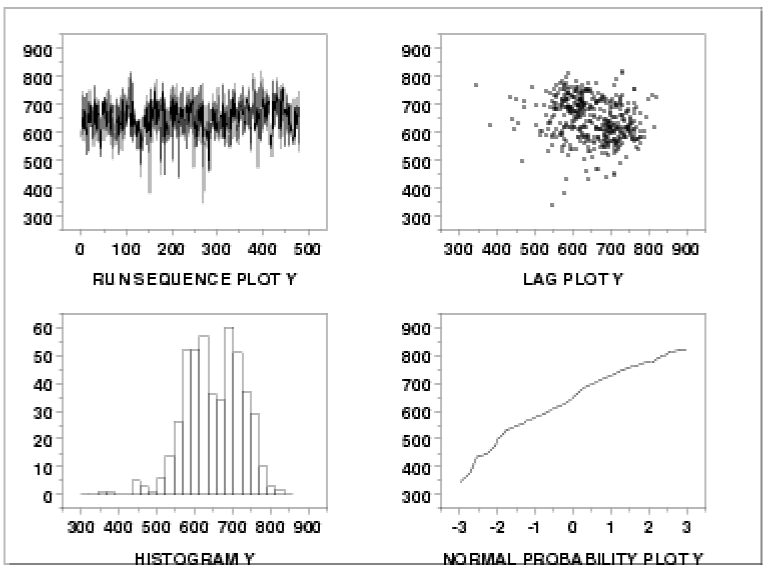
Numerical Summary

As a first step in the analysis, common summary statistics are computed for the response variable.

```
Sample size  = 480
Mean         =   650.0773
Median       =   646.6275
Minimum      =   345.2940
Maximum      =   821.6540
Range        =   476.3600
Stan. Dev.   =    74.6383
```

4-Plot

The next step is generate a 4-plot of the response variable.



This 4-plot shows:

1. The run sequence plot (upper left corner) shows that the location and scale are relatively constant. It also shows a few outliers on the low side. Most of the points are in the range 500 to 750. However, there are about half a dozen points in the 300 to 450 range that may require special attention.

   A run sequence plot is useful for designed experiments in that it can reveal time effects. Time is normally a nuisance factor. That is, the time order on which runs are made should not have a significant effect on the response. If a time effect does appear to exist, this means that there is a potential bias in the experiment that needs to be investigated and resolved.

2. The lag plot (the upper right corner) does not show any significant structure. This is another tool for detecting any potential time effect.
3. The histogram (the lower left corner) shows the response appears to be reasonably symmetric, but with a bimodal distribution.
4. The normal probability plot (the lower right corner) shows some curvature indicating that distributions other than the normal may provide a better fit.

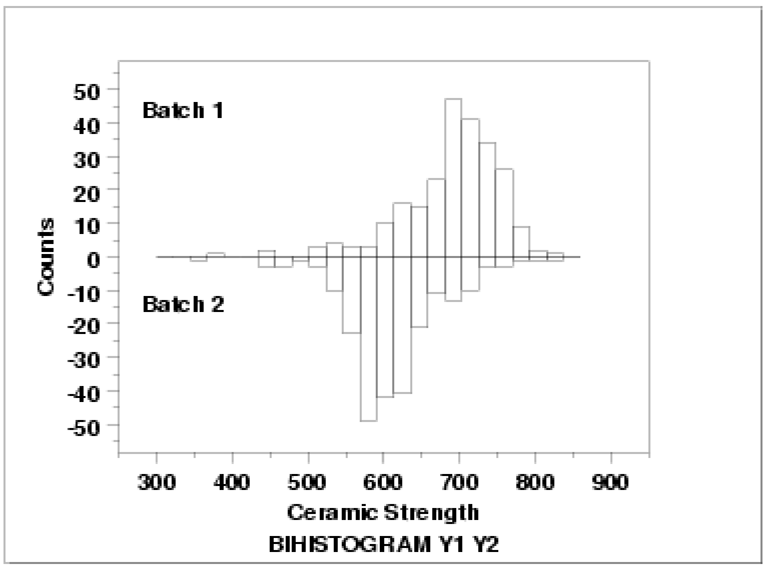
1.4.2.10.3. Analysis of the Batch Effect

Batch is a Nuisance Factor

The two nuisance factors in this experiment are the batch number and the lab. There are two batches and eight labs. Ideally, these factors will have minimal effect on the response variable.

We will investigate the batch factor first.

Bihistogram



This bihistogram shows the following.

1. There does appear to be a batch effect.

2. The batch 1 responses are centered at 700 while the batch 2 responses are centered at 625. That is, the batch effect is approximately 75 units.

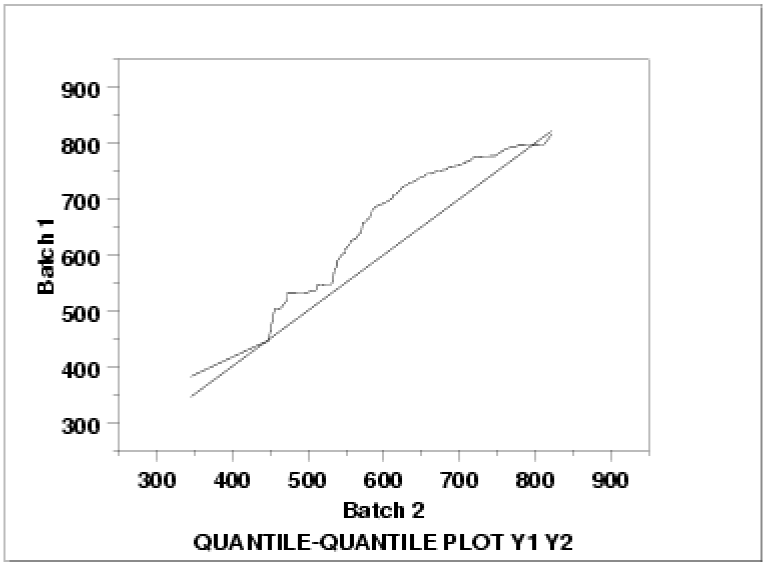
3. The variability is comparable for the 2 batches.

4. Batch 1 has some skewness in the lower tail. Batch 2 has some skewness in the center of the distribution, but not as much in the tails compared to batch 1.

5. Both batches have a few low-lying points.

Although we could stop with the bihistogram, we will show a few other commonly used two-sample graphical techniques for comparison.

Quantile-Quantile Plot

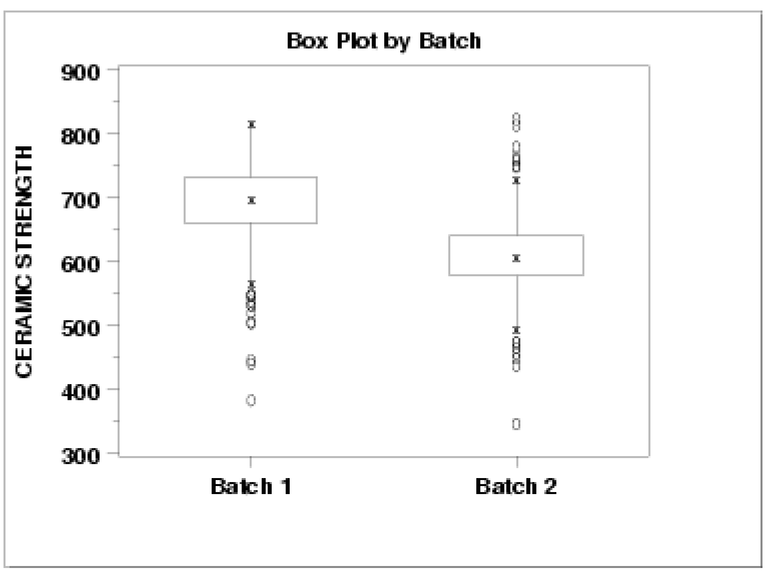


This q-q plot shows the following.

1. Except for a few points in the right tail, the batch 1 values have higher quantiles than the batch 2 values. This implies that batch 1 has a greater location value than batch 2.

2. The q-q plot is not linear. This implies that the difference between the batches is not explained simply by a shift in location. That is, the variation and/or skewness varies as well. From the bihistogram, it appears that the skewness in batch 2 is the most likely explanation for the non-linearity in the q-q plot.

Box Plot



This box plot shows the following.

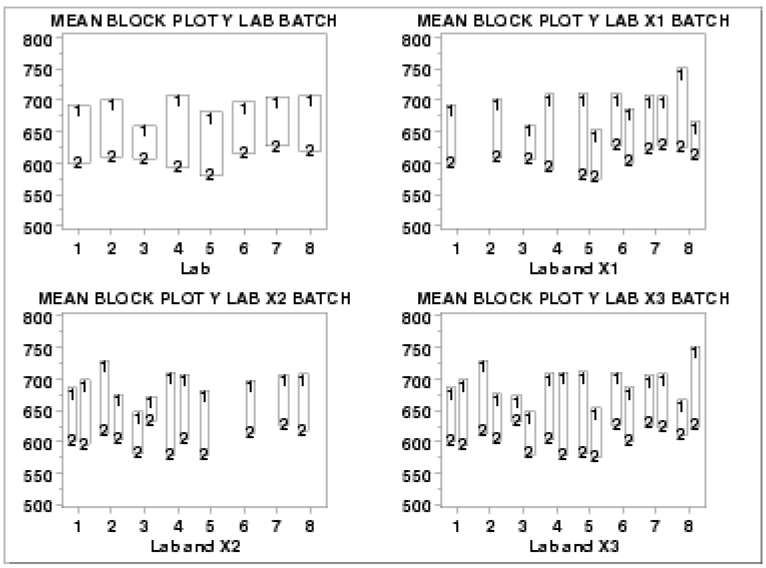
1. The median for batch 1 is approximately 700 while the median for batch 2 is approximately 600.

2. The spread is reasonably similar for both batches, maybe slightly larger for batch 1.

3. Both batches have a number of outliers on the low side. Batch 2 also has a few outliers on the high side. Box plots are a particularly effective method for identifying the presence of outliers.

Block Plots

A block plot is generated for each of the eight labs, with "1" and "2" denoting the batch numbers. In the first plot, we do not include any of the primary factors. The next 3 block plots include one of the primary factors. Note that each of the 3 primary factors (table speed = X1, down feed rate = X2, wheel grit size = X3) has 2 levels. With 8 labs and 2 levels for the primary factor, we would expect 16 separate blocks on these plots. The fact that some of these blocks are missing indicates that some of the combinations of lab and primary factor are empty.



These [block plots](#) show the following.

1. The mean for batch 1 is greater than the mean for batch 2 in **all** of the cases above. This is strong evidence that the batch effect is real and consistent across labs and primary factors.

Quantitative Techniques

We can confirm some of the conclusions drawn from the above graphics by using quantitative techniques. The *F*-test can be used to test whether or not the variances from the two batches are equal and the two sample *t*-test can be used to test whether or not the means from the two batches are equal. Summary statistics for each batch are shown below.

```
Batch 1:
    NUMBER OF OBSERVATIONS =  240
    MEAN                    =  688.9987
```

```
    STANDARD DEVIATION     =   65.5491
    VARIANCE               = 4296.6845

Batch 2:
    NUMBER OF OBSERVATIONS =  240
    MEAN                   =  611.1559
    STANDARD DEVIATION     =   61.8543
    VARIANCE               = 3825.9544
```

F-Test

The two-sided *F*-test indicates that the variances for the two batches are not significantly different at the 5 % level.

H_0: $\sigma_1^2 = \sigma_2^2$
H_a: $\sigma_1^2 \ne \sigma_2^2$

Test statistic: $F = 1.123$
Numerator degrees of freedom: $\nu_1 = 239$
Denominator degrees of freedom: $\nu_2 = 239$
Significance level: $\alpha = 0.05$
Critical values: $F_{1-\alpha/2,\nu_1,\nu_2} = 0.845$
$F_{\alpha/2,\nu_1,\nu_2} = 1.289$
Critical region: Reject H_0 if $F < 0.845$ or $F > 1.289$

Two Sample t-Test

Since the *F*-test indicates that the two batch variances are equal, we can pool the variances for the two-sided, two-sample *t*-test to compare batch means.

H_0: $\mu_1 = \mu_2$
H_a: $\mu_1 \ne \mu_2$

Test statistic: $T = 13.3806$
Pooled standard deviation: $s_p = 63.7285$
Degrees of freedom: $\nu = 478$
Significance level: $\alpha = 0.05$
Critical value: $t_{1-\alpha/2,\nu} = 1.965$
Critical region: Reject H_0 if $|T| > 1.965$

The *t*-test indicates that the mean for batch 1 is larger than the mean for batch 2 at the 5 % significance level.

Conclusions

We can draw the following conclusions from the above analysis.

1. There is in fact a significant batch effect. This batch effect is consistent across labs and primary factors.

2. The magnitude of the difference is on the order of 75 to 100 (with batch 2 being smaller than batch 1). The standard deviations do not appear to be significantly different.

3. There is some skewness in the batches.

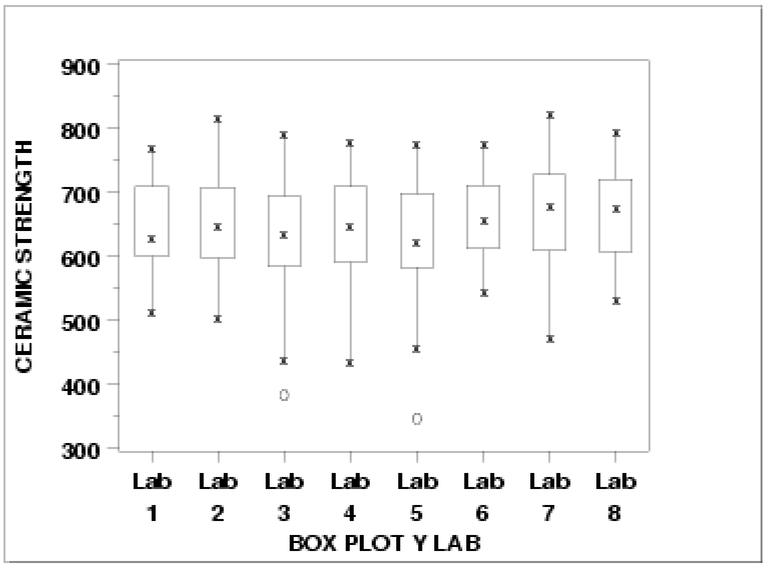
This batch effect was completely unexpected by the scientific investigators in this study.

Note that although the quantitative techniques support the conclusions of unequal means and equal standard deviations, they do not show the more subtle features of the data such as the presence of outliers and the skewness of the batch 2 data.

1.4.2.10.4. Analysis of the Lab Effect

Box Plot

The next matter is to determine if there is a lab effect. The first step is to generate a box plot for the ceramic strength based on the lab.

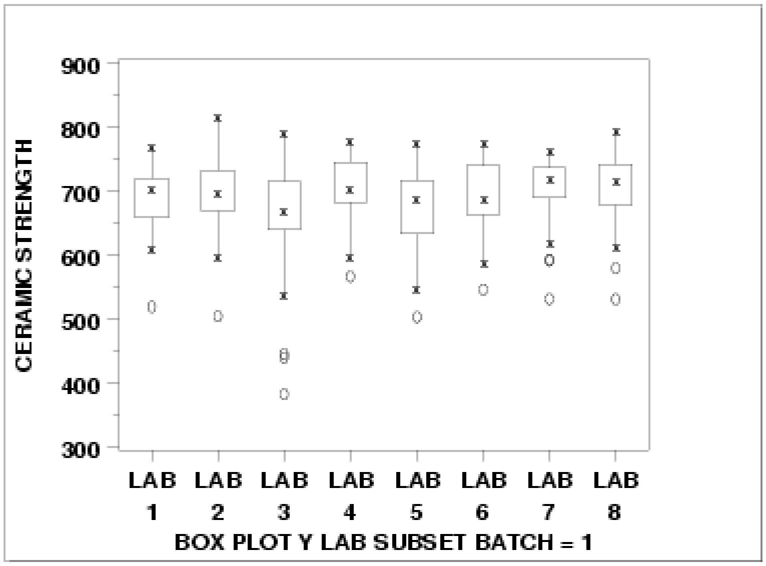


This box plot shows the following.

1. There is minor variation in the medians for the 8 labs.
2. The scales are relatively constant for the labs.
3. Two of the labs (3 and 5) have outliers on the low side.

Box Plot for Batch 1

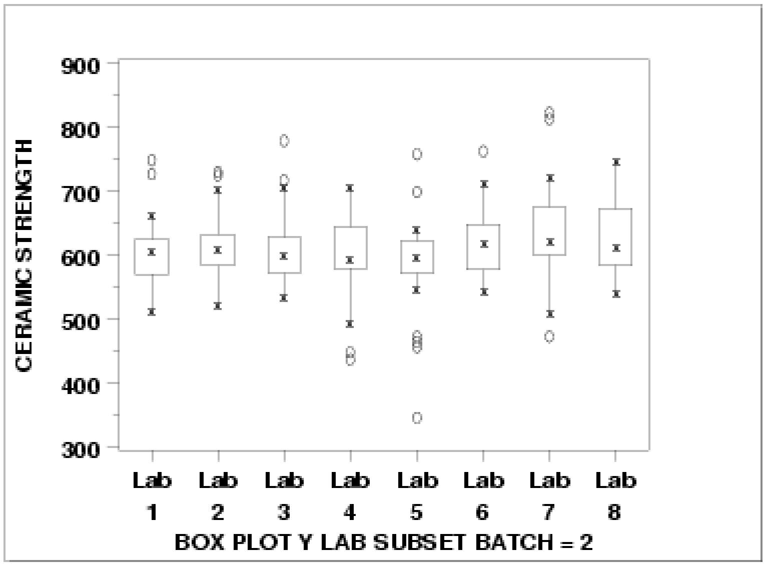
Given that the previous section showed a distinct batch effect, the next step is to generate the box plots for the two batches separately.



This box plot shows the following.

1. Each of the labs has a median in the 650 to 700 range.
2. The variability is relatively constant across the labs.
3. Each of the labs has at least one outlier on the low side.

Box Plot for Batch 2



This box plot shows the following.

1. The medians are in the range 550 to 600.
2. There is a bit more variability, across the labs, for batch2 compared to batch 1.
3. Six of the eight labs show outliers on the high side. Three of the labs show outliers on the low side.

Conclusions We can draw the following conclusions about a possible lab effect from the above box plots.

1. The batch effect (of approximately 75 to 100 units) on location dominates any lab effects.

2. It is reasonable to treat the labs as homogeneous.

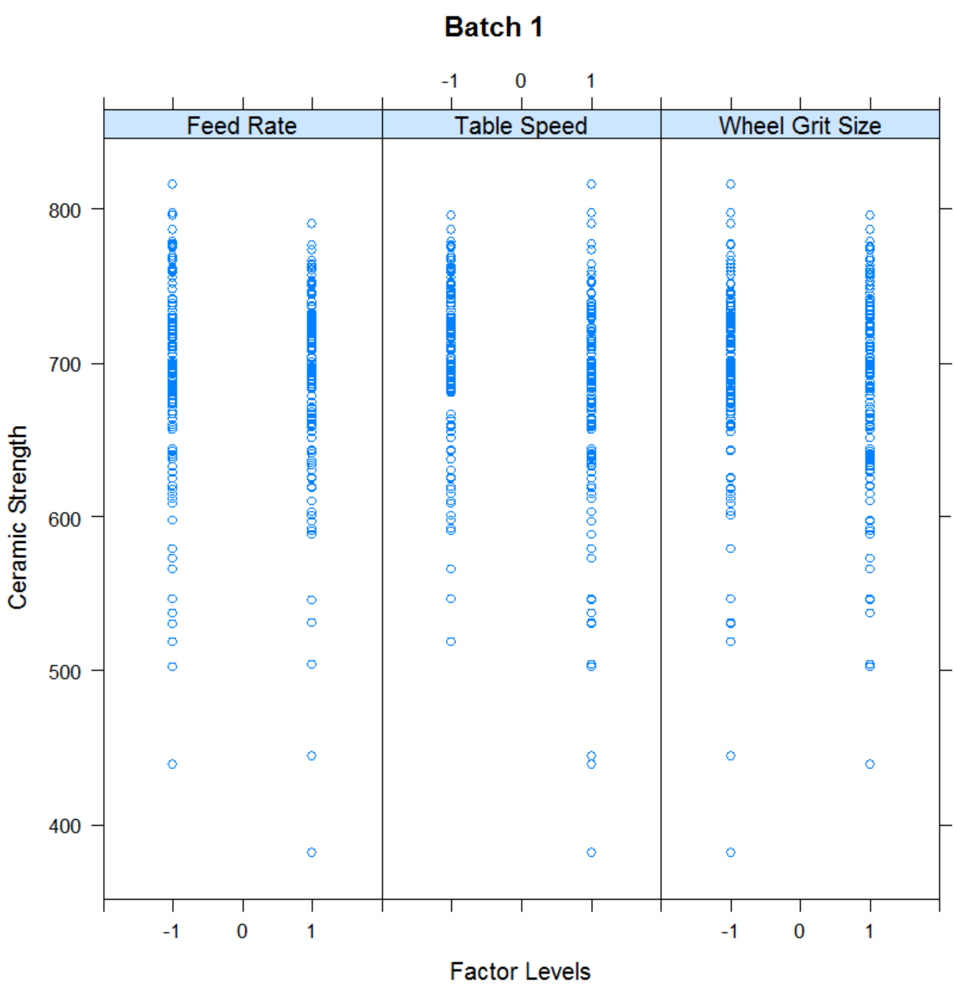
1.4.2.10.5. Analysis of Primary Factors

Main effects

The first step in analyzing the primary factors is to determine which factors are the most significant. The DOE scatter plot, DOE mean plot, and the DOE standard deviation plots will be the primary tools, with "DOE" being short for "design of experiments".

Since the previous pages showed a significant batch effect but a minimal lab effect, we will generate separate plots for batch 1 and batch 2. However, the labs will be treated as equivalent.

DOE Scatter Plot for Batch 1



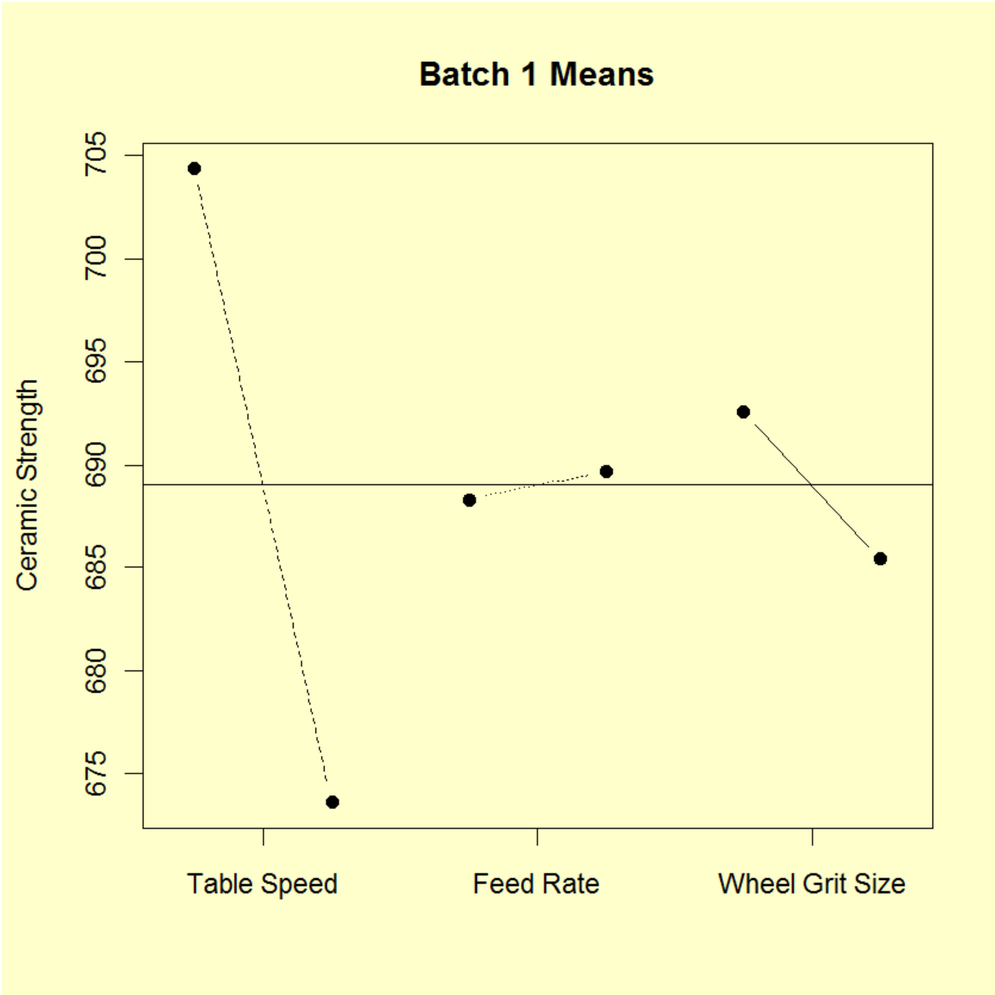
This DOE scatter plot shows the following for batch 1.

1. Most of the points are between 500 and 800.

2. There are about a dozen or so points between 300 and 500.

3. Except for the outliers on the low side (i.e., the points between 300 and 500), the distribution of the points is comparable for the 3 primary factors in terms of location and spread.

DOE Mean Plot for Batch 1



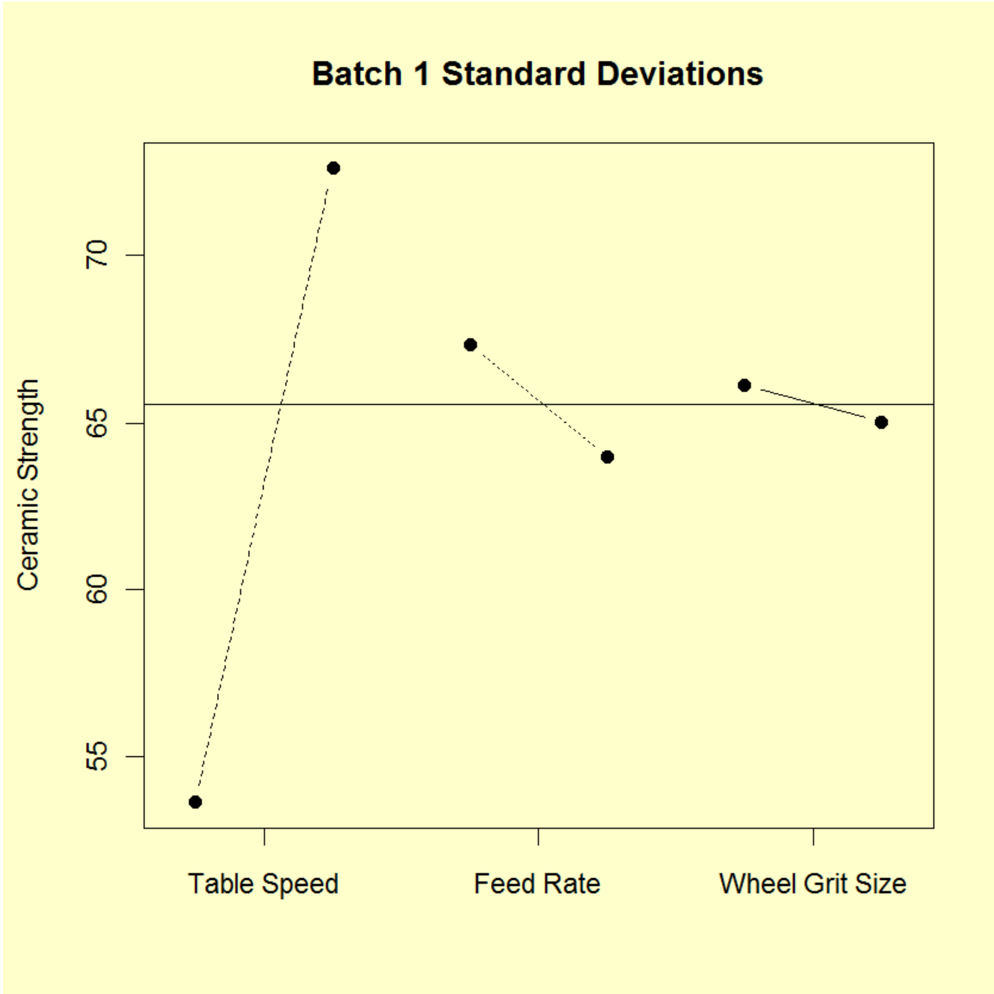
This DOE mean plot shows the following for batch 1.

1. The table speed factor (X1) is the most significant factor with an effect, the difference between the two points, of approximately 35 units.

2. The wheel grit factor (X3) is the next most significant factor with an effect of approximately 10 units.

3. The feed rate factor (X2) has minimal effect.

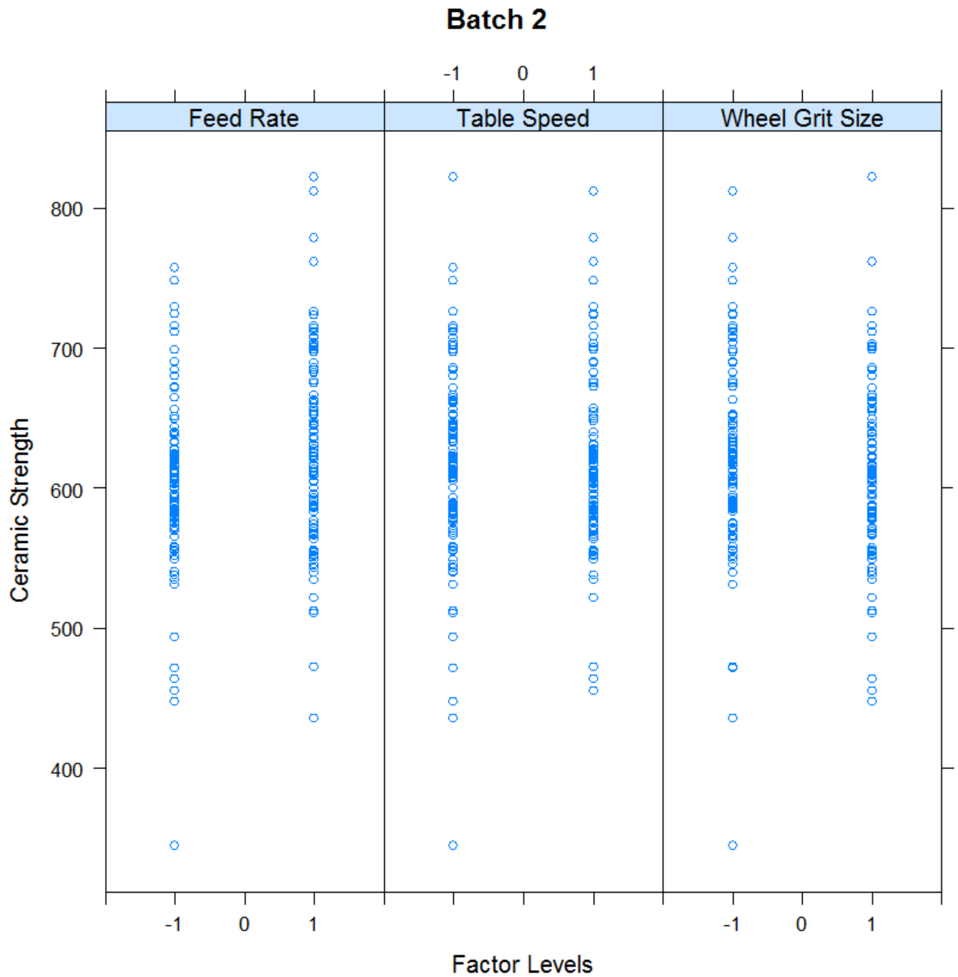
DOE SD Plot for Batch 1



This DOE standard deviation plot shows the following for batch 1.

1. The table speed factor (X1) has a significant difference in variability between the levels of the factor. The difference is approximately 20 units.
2. The wheel grit factor (X3) and the feed rate factor (X2) have minimal differences in variability.

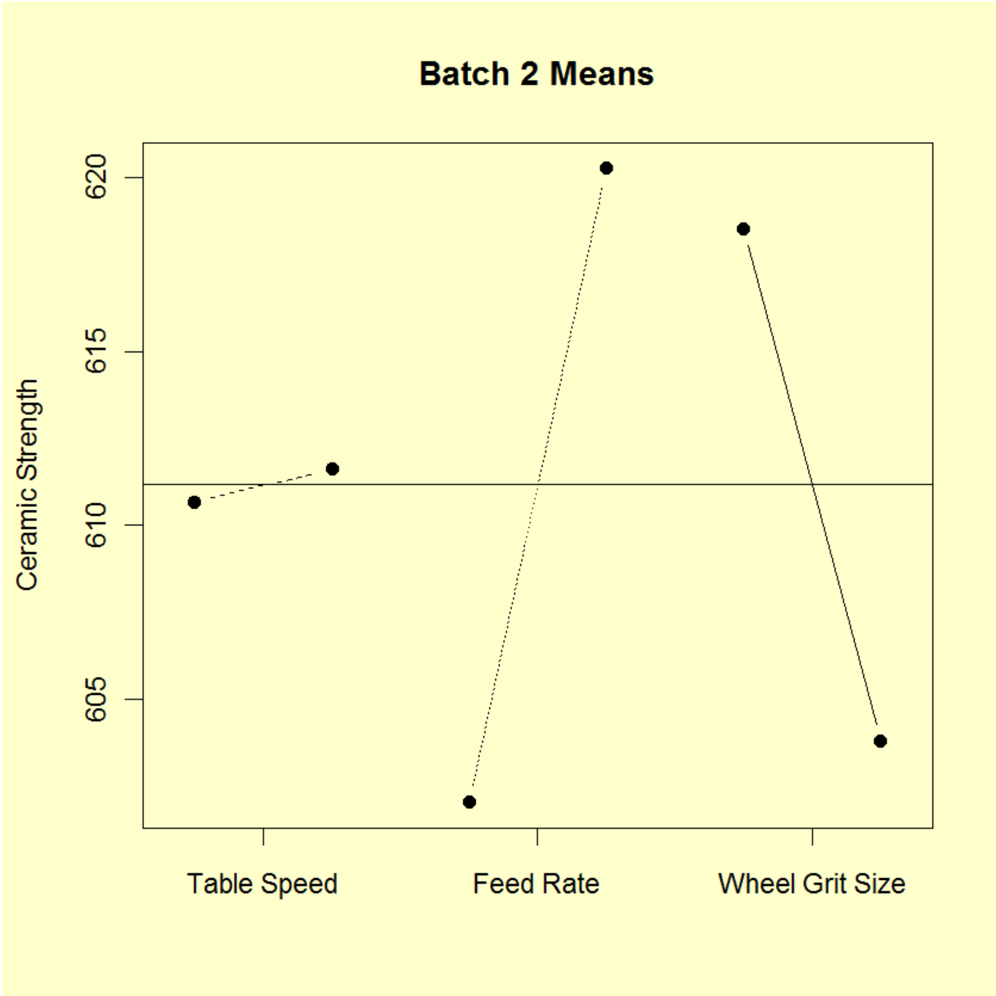
DOE Scatter Plot for Batch 2



This DOE scatter plot shows the following for batch 2.

1. Most of the points are between 450 and 750.
2. There are a few outliers on both the low side and the high side.
3. Except for the outliers (i.e., the points less than 450 or greater than 750), the distribution of the points is comparable for the 3 primary factors in terms of location and spread.

DOE Mean Plot for Batch 2

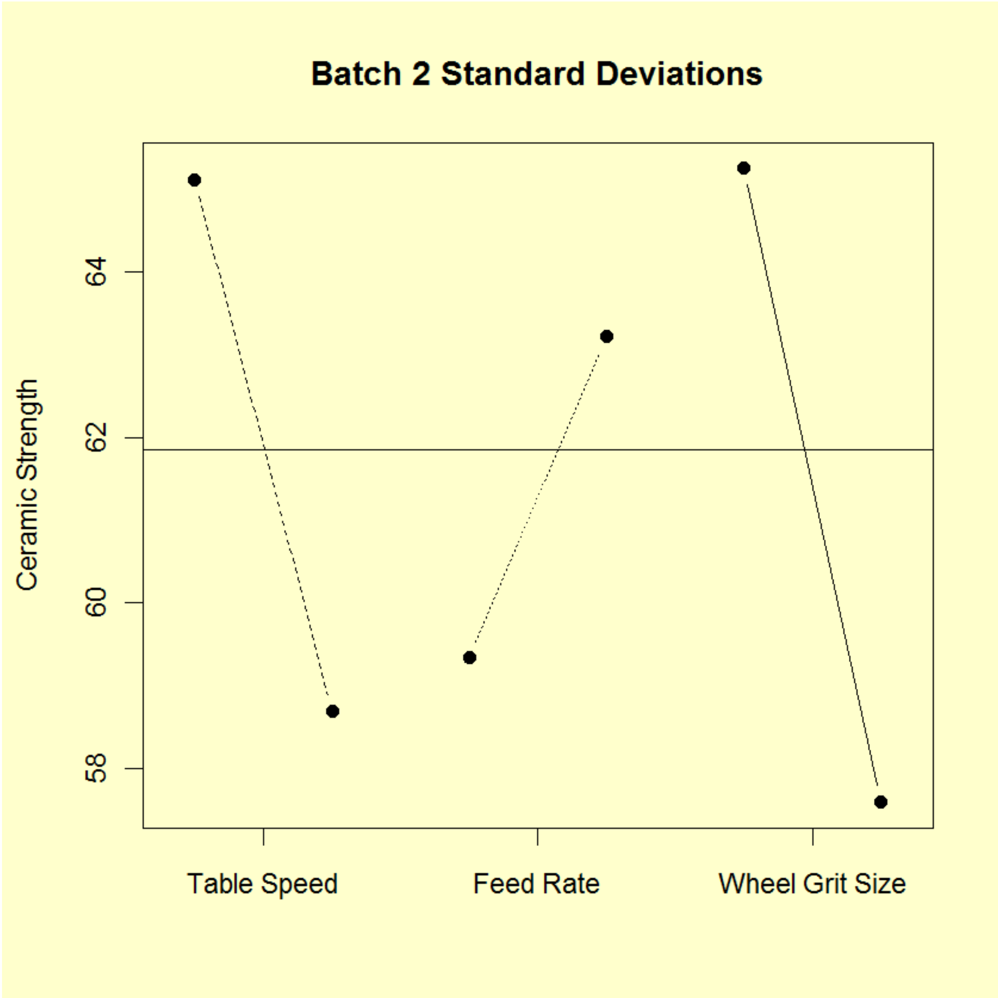


This DOE mean plot shows the following for batch 2.

1. The feed rate (X2) and wheel grit (X3) factors have an approximately equal effect of about 15 or 20 units.

2. The table speed factor (X1) has a minimal effect.

DOE SD Plot for Batch 2



This DOE standard deviation plot shows the following for batch 2.

1. The difference in the standard deviations is roughly comparable for the three factors (slightly less for the feed rate factor).

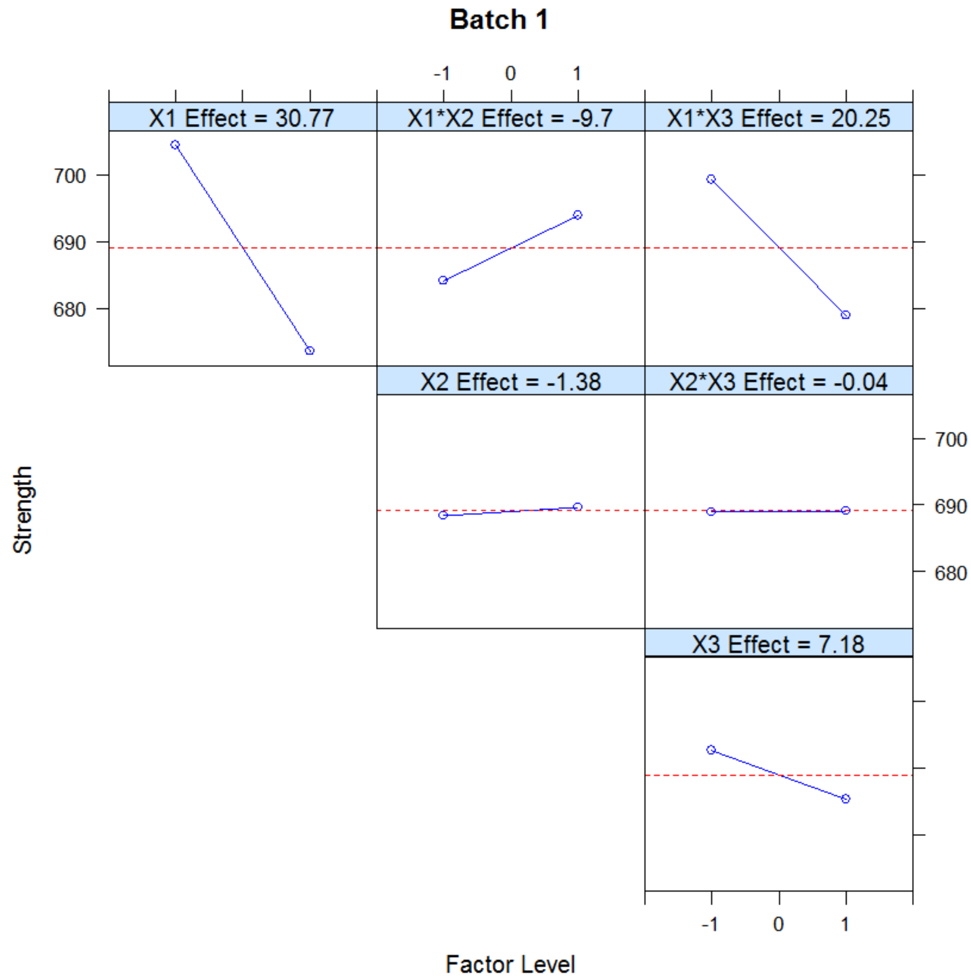
Interaction Effects

The above plots graphically show the main effects. An additonal concern is whether or not there any significant interaction effects.

Main effects and 2-term interaction effects are discussed in the chapter on Process Improvement.

In the following DOE interaction plots, the labels on the plot give the variables and the estimated effect. For example, factor 1 is table speed and it has an estimated effect of 30.77 (it is actually -30.77 if the direction is taken into account).

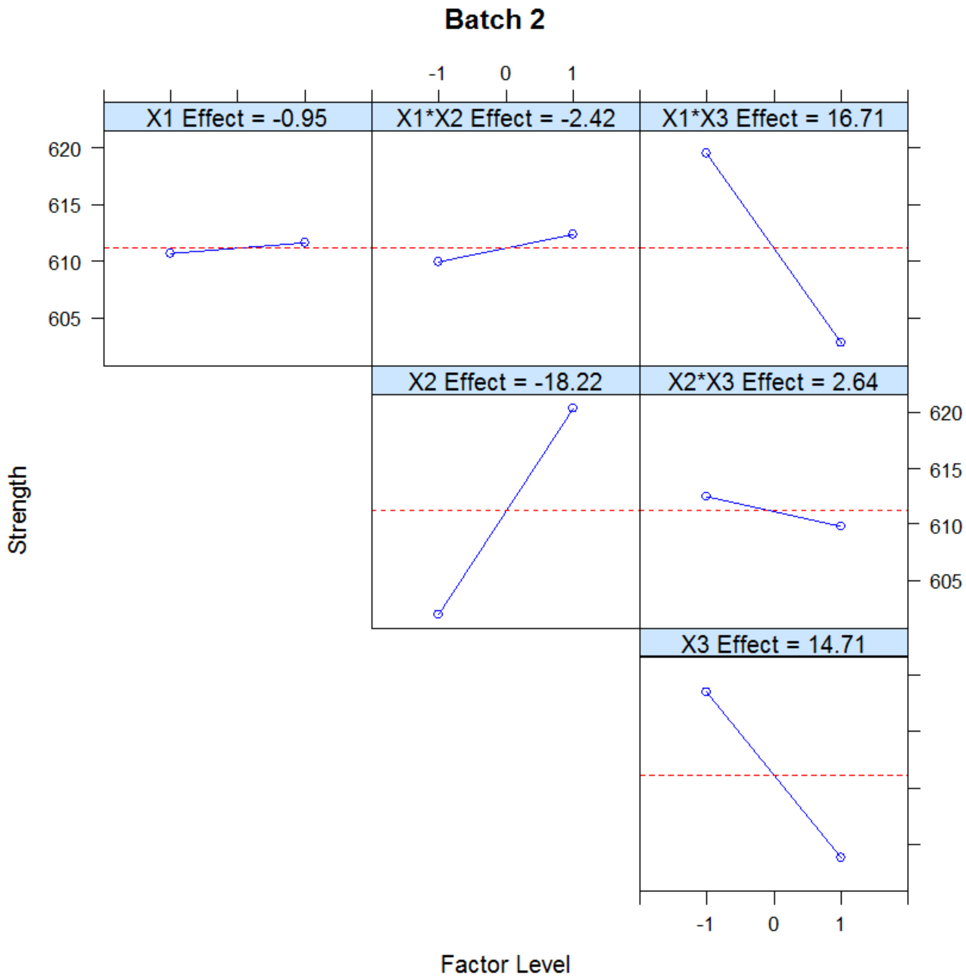
DOE Interaction Plot for Batch 1



The ranked list of factors for batch 1 is:

1. Table speed (X1) with an estimated effect of -30.77.
2. The interaction of table speed (X1) and wheel grit (X3) with an estimated effect of -20.25.
3. The interaction of table speed (X1) and feed rate (X2) with an estimated effect of 9.7.
4. Wheel grit (X3) with an estimated effect of -7.18.
5. Down feed (X2) and the down feed interaction with wheel grit (X3) are essentially zero.

DOE Interaction Plot for Batch 2



The ranked list of factors for batch 2 is:

1. Down feed (X2) with an estimated effect of 18.22.
2. The interaction of table speed (X1) and wheel grit (X3) with an estimated effect of -16.71.
3. Wheel grit (X3) with an estimated effect of -14.71
4. Remaining main effect and 2-factor interaction effects are essentially zero.

Conclusions

From the above plots, we can draw the following overall conclusions.

1. The batch effect (of approximately 75 units) is the dominant primary factor.
2. The most important factors differ from batch to batch. See the above text for the ranked list of factors with the estimated effects.

1.4.2.10.6. Work This Example Yourself

View Dataplot Macro for this Case Study

This page allows you to use Dataplot to repeat the analysis outlined in the case study description on the previous page. It is required that you have already downloaded and installed Dataplot and configured your browser. to run Dataplot. Output from each analysis step below will be displayed in one or more of the Dataplot windows. The four main windows are the Output window, the Graphics window, the Command History window, and the data sheet window. Across the top of the main windows there are menus for executing Dataplot commands. Across the bottom is a command entry window where commands can be typed in.

Data Analysis Steps	**Results and Conclusions**
Click on the links below to start Dataplot and run this case study yourself. Each step may use results from previous steps, so please be patient. Wait until the software verifies that the current step is complete before clicking on the next step.	*The links in this column will connect you with more detailed information about each analysis step from the case study description.*
1. Invoke Dataplot and read data. 1. Read in the data.	1. You have read 1 column of numbers into Dataplot, variable Y.
2. Plot of the response variable 1. Numerical summary of Y. 2. 4-plot of Y.	1. The summary shows the mean strength is 650.08 and the standard deviation of the strength is 74.64. 2. The 4-plot shows no drift in the location and scale and a

	bimodal distribution.
3. Determine if there is a batch effect. 1. Generate a bihistogram based on the 2 batches. 2. Generate a q-q plot. 3. Generate a box plot. 4. Generate block plots. 5. Perform a 2-sample t-test for equal means. 6. Perform an F-test for equal standard deviations.	1. The bihistogram shows a distinct batch effect of approximately 75 units. 2. The q-q plot shows that batch 1 and batch 2 do not come from a common distribution. 3. The box plot shows that there is a batch effect of approximately 75 to 100 units and there are some outliers. 4. The block plot shows that the batch effect is consistent across labs and levels of the primary factor. 5. The t-test confirms the batch effect with respect to the means. 6. The F-test does not indicate any significant batch effect with respect to the standard deviations.
4. Determine if there is a lab effect. 1. Generate a box plot for the labs with the 2 batches combined. 2. Generate a box plot for the labs for batch 1 only. 3. Generate a box plot for the labs for batch 2 only.	1. The box plot does not show a significant lab effect. 2. The box plot does not show a significant lab effect for batch 1. 3. The box plot does not show a significant lab effect for batch 2.
5. Analysis of primary factors. 1. Generate a DOE scatter plot for batch 1.	1. The DOE scatter plot shows the range of points and the

2. Generate a DOE mean plot for batch 1.	presence of outliers.
3. Generate a DOE sd plot for batch 1.	2. The DOE mean plot shows that table speed is the most significant factor for batch 1.
4. Generate a DOE scatter plot for batch 2.	3. The DOE sd plot shows that table speed has the most variability for batch 1.
5. Generate a DOE mean plot for batch 2.	4. The DOE scatter plot shows the range of the points and the presence of outliers.
6. Generate a DOE sd plot for batch 2.	5. The DOE mean plot shows that feed rate and wheel grit are the most significant factors for batch 2.
7. Generate a DOE interaction effects matrix plot for batch 1.	6. The DOE sd plot shows that the variability is comparable for all 3 factors for batch 2.
8. Generate a DOE interaction effects matrix plot for batch 2.	7. The DOE interaction effects matrix plot provides a ranked list of factors with the estimated effects.
	8. The DOE interaction effects matrix plot provides a ranked list of factors with the estimated effects.

1.4.3. References For Chapter 1: Exploratory Data Analysis

Anscombe, F. (1973), Graphs in Statistical Analysis, *The American Statistician*, pp. 195-199.

Anscombe, F. and Tukey, J. W. (1963), The Examination and Analysis of Residuals, *Technometrics*, pp. 141-160.

Barnett and Lewis (1994), *Outliers in Statistical Data*, 3rd. Ed., John Wiley and Sons.

Birnbaum, Z. W. and Saunders, S. C. (1958), A Statistical Model for Life-Length of Materials, *Journal of the American Statistical Association*, 53(281), pp. 151-160.

Bloomfield, Peter (1976), *Fourier Analysis of Time Series*, John Wiley and Sons.

Box, G. E. P. and Cox, D. R. (1964), An Analysis of Transformations, *Journal of the Royal Statistical Society*, pp. 211-243, discussion pp. 244-252.

Box, G. E. P., Hunter, W. G., and Hunter, J. S. (1978), *Statistics for Experimenters: An Introduction to Design, Data Analysis, and Model Building*, John Wiley and Sons.

Box, G. E. P., and Jenkins, G. (1976), *Time Series Analysis: Forecasting and Control*, Holden-Day.

Bradley, (1968). *Distribution-Free Statistical Tests*, Chapter 12.

Brown, M. B. and Forsythe, A. B. (1974), *Journal of the American Statistical Association*, 69, pp. 364-367.

Chakravarti, Laha, and Roy, (1967). *Handbook of Methods of Applied Statistics, Volume I*, John Wiley and Sons, pp. 392-394.

Chambers, John, William Cleveland, Beat Kleiner, and Paul Tukey, (1983), *Graphical Methods for Data Analysis*, Wadsworth.

Chatfield, C. (1989). *The Analysis of Time Series: An Introduction*, Fourth Edition, Chapman & Hall, New York, NY.

Cleveland, William (1985), *Elements of Graphing Data*, Wadsworth.

Cleveland, William and Marylyn McGill, Editors (1988), *Dynamic Graphics for Statistics*, Wadsworth.

Cleveland, William (1993), *Visualizing Data*, Hobart Press.

Devaney, Judy (1997), *Equation Discovery Through Global Self-Referenced Geometric Intervals and Machine Learning*, Ph.d thesis, George Mason University, Fairfax, VA.

Draper and Smith, (1981). *Applied Regression Analysis*, 2nd ed., John Wiley and Sons.

du Toit, Steyn, and Stumpf (1986), *Graphical Exploratory Data Analysis*, Springer-Verlag.

Efron and Gong (February 1983), A Leisurely Look at the Bootstrap, the Jackknife, and Cross Validation, *The American Statistician*.

Evans, Hastings, and Peacock (2000), *Statistical Distributions*, 3rd. Ed., John Wiley and Sons.

Everitt, Brian (1978), *Multivariate Techniques for Multivariate Data*, North-Holland.

Filliben, J. J. (February 1975), The Probability Plot Correlation Coefficient Test for Normality, *Technometrics*, pp. 111-117.

Fuller Jr., E. R., Frieman, S. W., Quinn, J. B., Quinn, G. D., and Carter, W. C. (1994), Fracture Mechanics Approach to the Design of Glass Aircraft Windows: A Case Study, *SPIE Proceedings*, Vol. 2286, (Society of Photo-Optical Instrumentation Engineers (SPIE), Bellingham, WA).

Gill, Lisa (April 1997), *Summary Analysis: High Performance Ceramics Experiment to Characterize the Effect of Grinding Parameters on Sintered Reaction Bonded Silicon Nitride, Reaction Bonded Silicon Nitride, and Sintered Silicon Nitride* , presented at the NIST - Ceramic Machining Consortium, 10th Program Review Meeting, April 10, 1997.

Granger and Hatanaka (1964), *Spectral Analysis of Economic Time Series*, Princeton University Press.

Grubbs, Frank (1950), Sample Criteria for Testing Outlying Observations, *Annals of Mathematical Statistics*, 21(1) pp. 27-58.

Grubbs, Frank (February 1969), Procedures for Detecting Outlying Observations in Samples, *Technometrics*, 11(1), pp. 1-21.

Hahn, G. J. and Meeker, W. Q. (1991), *Statistical Intervals*, John Wiley and Sons.

Harris, Robert L. (1996), *Information Graphics*, Management Graphics.

Hastie, T., Tibshirani, R. and Friedman, J. (2001), *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, Springer-Verlag, New York.

Hawkins, D. M. (1980), *Identification of Outliers*, Chapman and Hall.

Boris Iglewicz and David Hoaglin (1993), "Volume 16: How to Detect and Handle Outliers", *The ASQC Basic References in Quality Control: Statistical Techniques*, Edward F. Mykytka, Ph.D., Editor.

Jenkins and Watts, (1968), *Spectral Analysis and Its Applications*, Holden-Day.

Johnson, Kotz, and Balakrishnan, (1994), *Continuous Univariate Distributions, Volumes I and II*, 2nd. Ed., John Wiley and Sons.

Johnson, Kotz, and Kemp, (1992), *Univariate Discrete Distributions*, 2nd. Ed., John Wiley and Sons.

Kuo, Way and Pierson, Marcia Martens, Eds. (1993), *Quality Through Engineering Design"*, specifically, the article Filliben, Cetinkunt, Yu, and Dommenz (1993), *Exploratory Data Analysis Techniques as Applied to a High-Precision Turning Machine*, Elsevier, New York, pp. 199-223.

Levene, H. (1960). In *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling*, I. Olkin et al. eds., Stanford University Press, pp. 278-292.

McNeil, Donald (1977), *Interactive Data Analysis*, John Wiley and Sons.

Mendenhall, William and Reinmuth, James (1982), *Statistics for Management and Ecomonics, Fourth Edition*, Duxbury Press.

Mosteller, Frederick and Tukey, John (1977), *Data Analysis and Regression*, Addison-Wesley.

Natrella, Mary (1963), *Experimental Statistics*, National Bureau of Standards Handbook 91.

Nelson, Wayne (1982), *Applied Life Data Analysis*, Addison-Wesley.

Nelson, Wayne and Doganaksoy, Necip (1992), A Computer Program POWNOR for Fitting the Power-Normal and -Lognormal Models to Life or Strength Data from Specimens of Various Sizes, *NISTIR 4760*, U.S. Department of Commerce, National Institute of Standards and Technology.

Neter, Wasserman, and Kunter (1990). *Applied Linear Statistical Models*, 3rd ed., Irwin.

Pepi, John W., (1994), Failsafe Design of an All BK-7 Glass Aircraft Window, *SPIE Proceedings*, Vol. 2286, (Society of Photo-Optical Instrumentation Engineers (SPIE), Bellingham, WA).

The RAND Corporation (1955), *A Million Random Digits with 100,000 Normal Deviates*, Free Press.

Rosner, Bernard (May 1983), Percentage Points for a Generalized ESD Many-Outlier Procedure,*Technometrics*, 25(2), pp. 165-172.

Ryan, Thomas (1997), *Modern Regression Methods*, John Wiley.

Scott, David (1992), *Multivariate Density Estimation: Theory, Practice, and Visualization* , John Wiley and Sons.

Snedecor, George W. and Cochran, William G. (1989), *Statistical Methods*, Eighth Edition, Iowa State University Press.

Stefansky, W. (1972), Rejecting Outliers in Factorial Designs, *Technometrics*, 14, pp. 469-479.

Stephens, M. A. (1974). EDF Statistics for Goodness of Fit and Some Comparisons, *Journal of the American Statistical Association*, 69, pp. 730-737.

Stephens, M. A. (1976). Asymptotic Results for Goodness-of-Fit Statistics with Unknown Parameters, *Annals of Statistics*, 4, pp. 357-369.

Stephens, M. A. (1977). Goodness of Fit for the Extreme Value Distribution, *Biometrika*, 64, pp. 583-588.

Stephens, M. A. (1977). *Goodness of Fit with Special Reference to Tests for Exponentiality* , Technical Report No. 262, Department of Statistics, Stanford University, Stanford, CA.

Stephens, M. A. (1979). Tests of Fit for the Logistic Distribution Based on the Empirical Distribution Function, *Biometrika*, 66, pp. 591-595.

Tietjen and Moore (August 1972), Some Grubbs-Type Statistics for the Detection of Outliers, *Technometrics*, 14(3), pp. 583-597.

Tufte, Edward (1983), *The Visual Display of Quantitative Information*, Graphics Press.

Tukey, John (1977), *Exploratory Data Analysis*, Addison-Wesley.

Velleman, Paul and Hoaglin, David (1981), *The ABC's of EDA: Applications, Basics, and Computing of Exploratory Data Analysis*, Duxbury.

Wainer, Howard (1981), *Visual Revelations*, Copernicus.

Wilk, M. B. and Gnanadesikan, R. (1968), Probability Plotting Methods for the Analysis of Data, *Biometrika*, 5(5), pp. 1-19.